Spreading in a kinetic reaction-transport equation for population dynamics
CCSAMM

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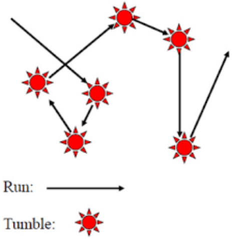
collaboration with E. Bouin (U. Paris-Dauphine)
work performed during my PhD at Université Lyon 1

Applications: *Escherichia coli*[1], *Rhinella marina*[2] (cane toad)



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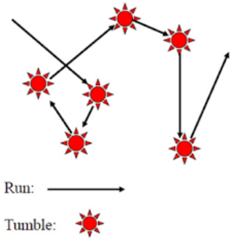

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- admissible velocity set: $V \subset \mathbb{R}^d$ (compact)

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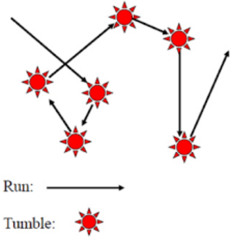

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- velocity redistribution: $M \in L^1(V)$ such that $\int_V vM(v)dv = 0$

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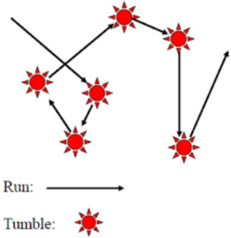

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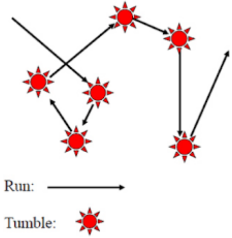

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- velocity redistribution: $M \in L^1(V)$ such that $\int_V vM(v)dv = 0$

+ Reproduction at rate $r > 0$ and intra-specific competition (KPP type)
+ homogeneous environment (No chemotactic effect !!!)

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Spreading in a kinetic reaction-transport equation

Chapman-Kolmogorov equation:

$$\partial_t f + v \cdot \nabla_x f = M(v)\rho - f + r\rho(M(v) - f), \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^d \times V. \tag{1}$$

Density: $\rho(t,x) := \int_V f(t,x,v)dv$

We will assume $V = B(0, v_{\max})$, $v_{\max} < +\infty$.

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How fast does the population spread in its environment?

First approach: Diffusion-approximation $(t, x, v) \to (\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, v)$, $r \to \varepsilon^2 r$

$$\varepsilon^2 \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = M\rho^\varepsilon - f^\varepsilon + \varepsilon^2 r \rho^\varepsilon (M - f^\varepsilon) \tag{2}$$

Assume $\int_V vM(v) = 0$ and $\int_V |v|^2 M(v) = \theta < +\infty$, then[3]

$$\lim_{\varepsilon \to 0} f^\varepsilon(t, x, v) = \rho(t, x) M(v), \tag{3}$$

where $\partial_t \rho - \theta \partial_{xx}^2 \rho = r\rho(1 - \rho)$ (Fisher-KPP).

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From Kolmogorov-Petrovskii-Piskunov ('37):

- travelling waves for all $c \geq 2\sqrt{r\theta}$
- compactly supported initial data: spreading at speed $2\sqrt{r\theta}$

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Spreading in a kinetic reaction-transport equation

This approach may underestimate the speed of propagation, due to:

- chemotactic effect (e.g *Escherichia coli*[4])
- strongly biased random walks (e.g *Rhinella marina*[5])

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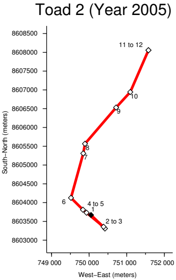

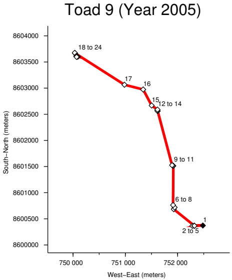

data: courtesy of G. Brown, B. Phillips, R. Shine

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Other approaches:

1. Travelling wave solutions
2. Hyperbolic limits

Travelling wave solutions

Looking for solution f of (1) of the form $f(t,x,v) = h(x \cdot e - ct, v)$:

$$(v \cdot e - c)\partial_1 h = M\rho_h - h + r\rho_h(M - h),$$

with $h(z,v) \to M(v)$ as $z \to -\infty$ and $h(z,v) \to 0$ as $z \to +\infty$

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$$(v \cdot e - c)\partial_1 \tilde{h} = M\rho_{\tilde{h}} - \tilde{h} + r\rho_{\tilde{h}} M.$$

(Note: supersolution)

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(Note: supersolution)
2. Exponential decay at the edge of the front: $\tilde{h}(z,v) \simeq e^{-\lambda z} Q(v)$.
Formally, we get the spectral problem: Find $(c\lambda, Q_\lambda)$ such that

$$\lambda c Q_\lambda = (\lambda v \cdot e - 1)Q_\lambda + (r+1)M \int_V Q_\lambda(v)dv.$$

Travelling wave solutions

Solving the spectral problem gives the dispersion relation

$$(r+1)\int_V \frac{M(v)}{1+\lambda(c - v\cdot e)}dv = 1. \tag{4}$$

When, $d = 1$ and $M > 0$,

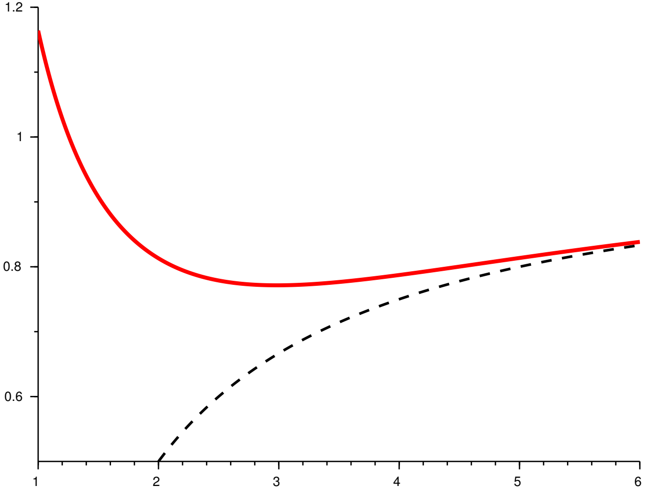
$$\lim_{c\to v_{\max - \frac{1}{\lambda}}} (r+1)\int_V \frac{M(v)}{1+\lambda(c - v\cdot e)}dv = +\infty.$$

Theorem (Bouin-Calvez-Nadin 2015)

Suppose $d = 1$ and $\inf_{v\in V} M(v) > 0$. Then, for $\lambda > 0$, there exists a unique $c(\lambda) > 0$ such that (4) holds. Moreover, there exist travelling wave solutions of (1) for all $c \geq c^* := \min_{\lambda > 0} c(\lambda)$.

Travelling wave solutions

$\lambda \mapsto c(\lambda)$ for $V = [-1, 1]$ and $M \equiv \frac{1}{2}$



Travelling wave solutions

Construction of travelling wave solutions for $c \geq c^*$:

- There exists a solution Q_λ to the spectral problem
- sub- and super-solution using Q_λ
- comparison principle

Non-existence of travelling wave solutions for $c < c^*$:

- relies on the fact that $c'(\lambda) = 0$

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Then, solution of the spectral problem is given $(v_{\max} - 1/\lambda, \mu)$, where

$$\mu = \left(1 - \frac{(r+1)}{\lambda} \int_V \frac{M(v)}{v_{\max} - v' \cdot e} dv'\right) \delta_{v_{\max} e} + \frac{dv}{\lambda(v_{\max} - v \cdot e)}$$

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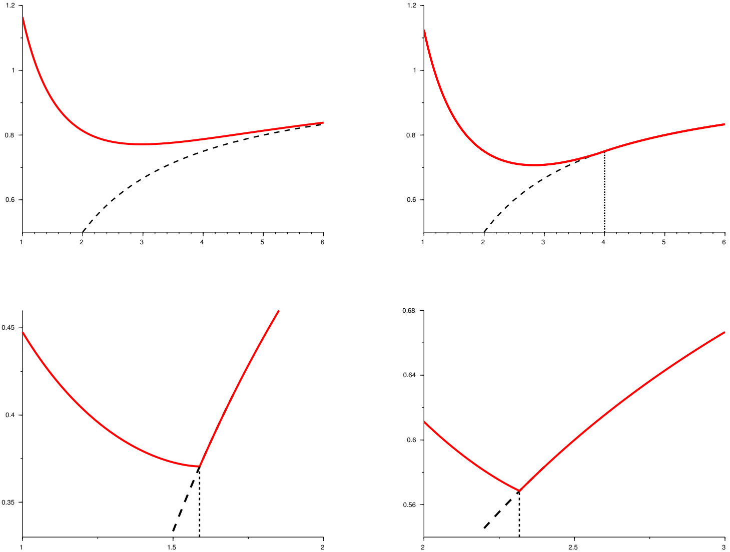
Occurs in the most simple cases: for $d = 2$, $V = D(0,1)$ and $M \equiv \frac{1}{\pi}$,

$$\frac{r+1}{\lambda} \int_V \frac{M(v)}{v_{\max} - v \cdot e} dv = \frac{r+1}{2\lambda}.$$

Travelling wave solutions



Travelling wave solutions

Result for general d and possibly vanishing M:

Theorem (Bouin, NC, submitted (2017))

Under previous assumptions, there exist travelling wave solutions for all $c \geq c^* = \min_{\lambda > 0} c(\lambda)$.

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Existence for $c \geq c^*$: exactly as in Bouin Calvez & Nadin's paper (sub- and super-solution)
Non-existence when $c < c^*$: requires more work (Hyperbolic limits)

Hyperbolic limits

Study (1) in the hyperbolic scale: $(t, x, v) \to (\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v)$.

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon}(M\rho^\varepsilon - f^\varepsilon) + \frac{r}{\varepsilon}\rho^\varepsilon(M - f^\varepsilon).$$

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WKB Ansatz: $\varphi^\varepsilon := -\varepsilon \ln(\frac{f^\varepsilon}{M})$, equivalently $f^\varepsilon = Me^{-\frac{\varphi^\varepsilon}{\varepsilon}}$.

$$\partial_t \varphi^\varepsilon + v \cdot \nabla_x \varphi^\varepsilon + r = (1 + r) \int_V M(v')(1 - e^{\frac{\varphi^\varepsilon - \varphi'^\varepsilon}{\varepsilon}} dv') + r\rho^\varepsilon$$

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Suppose $\varphi^\varepsilon \to \varphi^0$, then (formally)

$$f^\varepsilon \to 0 \quad \text{on} \ \left\{\varphi^0 > 0\right\},$$

$$f^\varepsilon \to M \quad \text{on} \ \left\{\varphi^0 = 0\right\}.$$

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What is φ^0? Thanks to Lipschitz uniform bounds[6] on φ^ε, φ^0 should be independent of v.

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Let us assume that $\varphi^\varepsilon = \varphi^0 - \varepsilon \ln(Q) + \mathcal{O}(\varepsilon^2)$. Then, formally,

$$\partial_t \varphi^0 + v \cdot \nabla_x \varphi^0 + r = (1+r) \int_V M(v') \left(1 - \frac{Q(v')}{Q(v)} \right) dv' + r e^{-\frac{\varphi^0}{\varepsilon}} \int_V Q(v') dv'$$

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Let $p := \nabla_x \varphi^0$ and $H = r - \partial_t \varphi^0$. On $\{\varphi^0 > 0\}$,

$$Q(v) = \frac{1+r}{1 + H - v \cdot p} \int_V M(v') Q(v') dv',$$

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Theorem (Bouin 2015)

For $d=1$, $M>0$ and for $\varphi(0,x,v)=\varphi_0(x)$, the sequence $(\varphi^\varepsilon)_\varepsilon$ converges uniformly locally to φ^0 which is the viscosity solution of

$$\begin{cases} \min\left(\partial_t\varphi^0 + H(\nabla_x\varphi^0) + r, \varphi^0\right) = 0, \\ \varphi^0(0,\cdot) = \varphi_0, \end{cases}$$

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where H is implicitly defined by (6) when such H exists, and $H(p) = v_{\max}|p| - 1$ otherwise.

φ^0 is a viscosity super-solution : let $\psi \in C^1(\mathbb{R}_+ \times \mathbb{R}^d)$ such that $\varphi^0 - \psi$ has a global strict minimum at $(t^0, x^0) \in \mathbb{R}_+^* \times \mathbb{R}^d$.

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$$\text{Let us show that } \frac{\partial \psi}{\partial t}(t^0, x^0) + H(\nabla_x \psi(t^0, x^0)) + r \geq 0.$$

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$\varphi^\varepsilon - \psi^\varepsilon$ has a minimum at $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$.

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&= (1+r)\left(1 - \frac{1}{Q}\int_V M' Q' dv'\right) + r \int_V e^{-\frac{\varphi'^\varepsilon}{\varepsilon}} dv'
\end{aligned}$$

1st case: Then, $(1+r)\int_V M(v)Q(v)dv = 1$ and $Q \in L^\infty(V)$
$\implies (t^\varepsilon, x^\varepsilon) \to (t^0, x^0)$, $v^\varepsilon \to v^*$
At $(t^\varepsilon, x^\varepsilon, v^\varepsilon)$,

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Take the limit $\varepsilon \to 0$:

$$\frac{\partial \psi}{\partial t}(t^0, x^0) + v^* \cdot \nabla_x \psi(t^0, x^0) + r \geq -H(\nabla_x \psi(t^0, x^0)) + v^* \cdot \nabla_x \psi(t^0, x^0)$$

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Q_K is bounded: same procedure, then take $K \to +\infty$

Hyperbolic limits

Back to spreading issues:
Let f be a travelling wave solution: $f(t, x, v) = h(x \cdot e - ct, v)$ and $f^\varepsilon = h(\frac{x \cdot e - ct}{\varepsilon}, v)$.

Hyperbolic limits

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3. Hopf-Lax formula: $\varphi^0(t,x) = \max\left(\min_{y\in\mathbb{R}^d}\left\{tL\left(\frac{x-y}{t}\right) + \varphi^0(0,y)\right\}, 0\right)$, where L is the Legendre transform of H: $L(p) = \sup_{q\in\mathbb{R}^d}\{p\cdot q - H(q)\}$

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5. Since $h(z,v) \underset{z\to-\infty}{\longrightarrow} M(v)$ and $h(z,v) \underset{z\to+\infty}{\longrightarrow} 0$, we have $c \geq c^*$.

Related works

- $r = 0$: Bouin-Calvez 2012 and NC 2017
- Unbounded velocity set (superlinear spreading): Bouin-Calvez-Grenier-Nadin 2016 (submitted)
- $r = 0$ and force terme: NC (work in progress)
- More general reaction terms (In 1D !!!): Bouin 2016.
- genetic trait structured population: Bouin-Mirrahimi (2015)

Aknowledgement

Thank you for your attention !

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