Diffusion-approximation for a kinetic equation with perturbed velocity redistribution process

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Abstract

We derive the hydrodynamic limit of a kinetic equation with a stochastic, short range perturbation of the velocity operator. Under some mixing hypotheses on the stochastic perturbation, we establish a diffusion-approximation result: the limit we obtain is a parabolic stochastic partial differential equation on the macroscopic parameter, the density here.

Keywords: diffusion-approximation, hydrodynamic limit, run-and-tumble

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Contents

1 Introduction

Let $\mathbb{T}^d$ denote the d-dimensional torus. Let V be a bounded domain of $\mathbb{R}^d$, say $V \subset \bar{B}_{\mathbb{R}^d}(0,1)$, and let ν be a probability measure on V. We consider the following kinetic random equation:

$$\partial_t f^\varepsilon + \frac{v}{\varepsilon} \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2}(M\rho^\varepsilon - f^\varepsilon) + \frac{1}{\varepsilon^2}\rho^\varepsilon v \cdot \nabla_x \bar{m}_t^\varepsilon, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{T}^d \times V, \tag{1.1}$$

with initial condition

$$f^\varepsilon(0) = f^\varepsilon_{\text{in}} \in L^2(\mathbb{T}^d \times V). \tag{1.2}$$

In (1.1), ρ^ε is the density associated to f^ε:

$$\rho^\varepsilon = \rho(f^\varepsilon) = \int_V f^\varepsilon(v) d\nu(v). \tag{1.3}$$

The parameter $\varepsilon > 0$ is small and we will study the limit of (1.1) when $\varepsilon \to 0$. The random character of (1.1) comes from the factor $v \cdot \nabla_x \bar{m}_t^\varepsilon$. In this term

$$\bar{m}_t^\varepsilon(x) = \bar{m}_{\varepsilon^{-2}t}(x), \tag{1.4}$$

where $(\bar{m}_t)$ is a stationary stochastic process over $C^3(\mathbb{T}^d)$. The function M is a probability density function on (V,ν). We will assume that M is bounded from above and from below:

$$\alpha \leq M(v) \leq \alpha^{-1}, \text{ for } \nu \text{ a.e. } v \in V, \tag{1.5}$$

where $\alpha \in (0,1)$. Due to (1.4), Equation (1.1) is obtained from the rescaling $f^\varepsilon(t,x,v) = f(\varepsilon^2 t, x, v)$, where f is solution to

$$\partial_t f + \varepsilon v \cdot \nabla_x f = (M\rho - f) + \rho v \cdot \nabla_x \bar{m}_t, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{T}^d \times V. \tag{1.6}$$

For $\varepsilon = 0$, (1.6) reduces to the equation

$$\partial_t f = (M\rho - f) + \rho v \cdot \nabla_x \bar{m}_t, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{T}^d \times V. \tag{1.7}$$

Under some mixing hypotheses on the process $(\bar{m}_t)$, Equation (1.7) has a unique invariant measure. This invariant measure is the law of a particular solution $\rho(x)\bar{M}_t$ (note that x is a parameter in (1.7)). This invariant solution is computed explicitly in Section 2.4, see (2.34). Consider the evolution given by (1.6) when the initial datum is close[1] to the equilibrium $\rho(x)\bar{M}_0(v)$. On the long time scale $\varepsilon^{-2}t$, we show that the rescaled unknown f^ε solution to (1.1) is close to a local equilibrium $\rho_t \bar{M}_{\varepsilon^{-2}t}$, and we give the evolution for the macroscopic parameter (ρ_t). The fact that $t \mapsto \varepsilon^{-2}t$ is the right time rescaling is due to the structure of (1.6) and to the following cancellation and normalization properties of M and (V,ν):

$$\int_V Q(v) d\nu(v) = 0, \quad \int_V M(v) d\nu(v) = 1, \tag{1.8}$$

where $Q(v) \in \{v_i, v_i M(v), v_i v_j v_k M(v)\}$, for $i,j,k \in \{1,\dots,d\}$. Our precise statement is the following one.

[1] actually, it is not necessary to start close to equilibrium, since the dynamics of (1.6) brings the solution close to local equilibrium in short time, see the bound on the entropy dissipation in (4.6)-(4.4)

Theorem 1.1. *Assume that $(\bar{m}_t)$ is an admissible pilot process in the sense of Definition 1.1. Assume that (M, V, ν) satisfy (1.5), (1.8). Let $f_{\mathrm{in}}^\varepsilon \in L^2(\mathbb{T}^d \times V)$ be a sequence of non-negative functions. Suppose also that*

$$\rho(f_{\mathrm{in}}^\varepsilon) \to \rho_{\mathrm{in}} \text{ in } L^2(\mathbb{T}^d), \quad \sup_{0<\varepsilon<1} \|f_{\mathrm{in}}^\varepsilon\|_{L^2(\mathbb{T}^d\times V)} \leq C_{\mathrm{in}} < +\infty. \tag{1.9}$$

Let f_t^ε be the solution to (1.1) with initial datum $f_{\mathrm{in}}^\varepsilon$. Let $\bar{M}_t^\varepsilon = \bar{M}_{\varepsilon^{-2}t}$ be the equilibrium given by (2.34). Then we have

$$\int_0^t \|f_s^\varepsilon - \rho(f_s^\varepsilon)\bar{M}_s^\varepsilon\|_{L^2(\mathbb{T}^d\times V)}^2 ds \leq \frac{C_{\mathrm{in}}^2 e^t}{\alpha^2}\varepsilon^2 \tag{1.10}$$

almost-surely, and, for all $\sigma > 0$, the convergence $\rho(f_t^\varepsilon) \to \rho_t$ in law on $C([0,T]; H^{-\sigma}(\mathbb{T}^d))$, where ρ is the solution to the stochastic partial differential equation

$$d\rho = \mathrm{div}(K^*\nabla_x\rho + \Psi\rho)dt + \sqrt{2}\,\mathrm{div}_x(\rho S^{1/2} dW(t)), \tag{1.11}$$

with initial condition $\rho(0) = \rho_{\mathrm{in}}$. In (1.11), $W(t)$ is a cylindrical Wiener process on $L^2(\mathbb{T}^d)$, S is the covariance operator defined by (3.29). The coefficients K^* and Ψ have the following expression:

$$K^* = K(M) + \mathbb{E}\left[(R_0R_1\chi)(\bar{m}_0) \otimes \chi(\bar{m}_0)\right], \quad K(M) := \int_V v \otimes v M(v) d\nu(v) \tag{1.12}$$

and

$$\Psi = \mathbb{E}\left[\mathrm{div}_x[\chi(\bar{m}_0)](R_0R_1\chi)(\bar{m}_0)\right], \tag{1.13}$$

where $\chi(n)$ and the resolvent R_α are defined in (1.25) and (1.24) respectively.

Remark 1.1 (Stratonovitch). The Stratonovitch form of (1.11) is

$$d\rho = \mathrm{div}(K_{\mathrm{Strato}}\nabla_x\rho + \Psi_{\mathrm{Strato}}\rho)dt + \sqrt{2}\,\mathrm{div}_x(\rho \circ S^{1/2} dW(t)), \tag{1.14}$$

where

$$K_{\mathrm{Strato}} = K(M) + \mathbb{E}\left[(R_1\chi)(\bar{m}_0) \otimes \chi(\bar{m}_0)\right], \quad \Psi_{\mathrm{Strato}} = \mathbb{E}\left[\mathrm{div}_x[\chi(\bar{m}_0)](R_1\chi)(\bar{m}_0)\right]. \tag{1.15}$$

Let us do the following comments about the result of Theorem 1.1.

1. In the deterministic case ($\bar{m}^\varepsilon \equiv 0$), f^ε converges to ρM, where ρ is the solution of the diffusion equation
$$\partial_t\rho - \mathrm{div}\left(K(M)\nabla_x\rho\right) = 0, \tag{1.16}$$
with initial condition $\rho(0) = \rho_{\mathrm{in}}$, see [11], for example, for a proof of this result (a probabilistic proof can also be found elsewhere, in [10, Section 5.1] for instance). We prove in Proposition 3.5 that, when $(\bar{m}_t)$ is reversible, we have $K^* \geq K$, in the sense that the matrix $K^* - K$ is non-negative. If we are only concerned with the convergence of the average $r^\varepsilon := \mathbb{E}\rho^\varepsilon$, we obtain that $r^\varepsilon \to r$ in $C([0,T]; H^{-\eta}(\mathbb{T}^d))$, where r is solution to
$$\partial_t r - \mathrm{div}\left(K^*\nabla_x r + \Psi r\right) = 0. \tag{1.17}$$
In that case, there is more diffusion in (1.17) than in (1.16).

2. The limit obtained in this paper should be compared to the approximation-diffusion results obtained in the context of hydrodynamic limits of kinetic equations in [9, 7, 10]. In the first two papers [9, 7] the order of the stochastic perturbation is weaker than here in (1.1) and, more precisely, the progression is the following one: in [9], the perturbed test-function method of [19], developed in the context of ordinary differential equation is combined with the deterministic hydrodynamic limit. In [7], tools for strong convergence are developed and non-linear equations are treated. In [10], more singular problems (more singular in the sense that the equilibria of the unperturbed equation are stochastic, not deterministic) are considered, in a linear setting however. Here also we consider a singular situation in a linear setting, a framework which is very close to the one considered in [10]. A noticeable difference with [10] is the fact that the space V of velocity is bounded here, while in [10], V is the whole space $\mathbb{R}^d$. As a consequence, we are able to show (1.10) thanks to a relative entropy estimate, a procedure which is not working for the time being for the problem considered in [10], due to some moments in v that cannot be controlled.

3. Diffusion-approximation for PDEs has been studied by Pardoux and Piatnitski [20], in the context of stochastic homogenization of parabolic equations, by Marty, De Bouard, Debussche, Gazeau, Tsutsumi [18, 5, 8, 6] for Schrödinger equations and by Bal, Fouque, Garnier, Papanicolaou, Sølna and their co-authors (see [1, 13, 14] for example) for propagation of waves in random media.

4. Let us set temporarily $N(t,x) = \bar{m}_t(x)$: $N(t,x)$ will stand for the concentration of a chemotactic substance at time t at point x in a model of evolution of a cell due to a run-and-tumble process. There are two alternative steps here thus. A phase of run, with an evolution at constant speed V, which corresponds to the evolution equation $\dot{X}(t) = V$. A tumble phase, given by a redistribution, after a random time with exponential law, of the velocity V. After rescaling, Equation (1.1) or (1.6) is the evolution equation for the density of the law of the resulting process (X_t, V_t) when the redistribution of velocity is done according to the following scheme: the new velocity is chosen independently from the previous one, according to a law with density $\check{M}\colon v \mapsto M(v) + v \cdot \nabla_x N$ with respect to ν. We see $\check{M}$ as a perturbation of M, random since N is random. This accounts for the title of the paper. Note also that it is understood that the perturbation $v \cdot \nabla_x N$ gives more weight to velocities that drive the organism under consideration towards zone with higher concentration of chemotactic substance. This is conceivable if the organism is big enough to be sensitive to gradients of the chemotactic substance at its own scale.

The organization of the paper is the following one. In the following Section 1.1, we describe precisely the class of driving random term $(\bar{m}_t)$ that we consider. In Section 2, we study Equation (1.1) at fixed ε and the ergodic properties of Equation (1.7). In Section 3, we use the perturbed test-function method to identify the limit generator that arises when $\varepsilon \to 0$. Then, in Section 4, we show that (ρ^ε) is tight and converges in law to the solution of the martingale problem associated to the limit generator $\mathscr{L}$. In the last Section 5, we show that this limit is the law of the solution to the stochastic PDE (1.11).

There are some standards facts about processes which are usually taken for granted without expanding too much on their proof. Since we consider processes with infinite-dimensional state spaces, these proofs, with all their details, can be quite lengthy. In [10], such complete proofs have been given, with the aim to be used for future reference. We will use and make reference to [10] here, when needed. For example, we do not give the proof of Theorem 2.4 or Corollary 2.7, or all the details of Section 5.2.

1.1 The driving random term

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $F = C^3(\mathbb{T}^d)$ be the Banach space with norm

$$\|m\|_F = \sup_{x\in\mathbb{T}^d, 0\leq k\leq 3} \left[|D^k m(x)|\right].$$

This is the state space for the process $(\bar{m}_t)$: we consider a stationary, homogeneous, càdlàg Markov process $(\bar{m}_t)_{t\geq 0}$ over F with generator A (the generator is defined according to the theory developed by Priola in [21], see also [10, Appendix B]). Let $P(t, n, B)$ be a transition function for $(\bar{m}_t)$ associated to the filtration generated by $(\bar{m}_t)$ (see, e.g., [12, p. 156] for the definition), satisfying the Chapman-Kolmogorov relation

$$P(t+s, n, B) = \int_F P(s, m, B) dP(t, n, dm), \tag{1.18}$$

for all $s, t \geq 0$, $n \in F$, B Borel subset of F. Up to a modification of the probability space, and by identification of versions of the processes, we are given processes $m(t, s; n_0)$ with transition function P satisfying $\mathbb{P}(m(s, s; n_0) \in B) = \mu(B)$ where μ, a Borel probability measure on F, is the law of n_0. We can also assume that $(\bar{m}_t)$ is defined for all $t \in \mathbb{R}$ (see the beginning of [10, Section 2] for the justification of these assertions).

Definition 1.1 (Admissible pilot process)**.** Let $(\bar{m}_t)_{t\geq 0}$ be a càdlàg, stationary, homogeneous Markov process of generator A over F. We say that $(\bar{m}_t)_{t\geq 0}$ is an admissible pilot process if the conditions (1.19), (1.20), (1.21), (1.23), (1.28) below are satisfied.

Our first hypothesis is that there exists a stable ball: there exists $\mathtt{R} \geq 0$ such that: almost-surely, for all n with $\|n\|_F \leq \mathtt{R}$, for all $t \geq 0$,

$$\|m(t; n)\|_F \leq \mathtt{R}. \tag{1.19}$$

We will assume that $\mathtt{R}$ is sufficiently small in order to ensure that the matrix K^* defined by (1.12) is positive. We suppose therefore that

$$\mathtt{R} \leq \frac{\alpha}{4}. \tag{1.20}$$

Our second hypothesis is about the law λ of $\bar{m}_t$. We assume that it is supported in the ball $\bar{B}_{\mathtt{R}}$ of F (therefore, it has moments of all orders) and that it is centred:

$$\int_F n \, d\lambda(n) = \mathbb{E}\left[\bar{m}_t\right] = 0, \tag{1.21}$$

for all $t \geq 0$. Note that a consequence of this hypothesis is that: almost-surely, for all $t \geq 0$,

$$\|\bar{m}_t\|_F \leq \mathtt{R}. \tag{1.22}$$

Our third hypothesis is a mixing hypothesis: we assume that there exists a continuous, non-increasing, positive and integrable function $\gamma_{\mathrm{mix}} \in L^1(\mathbb{R}_+)$ such that, for all probability measures μ, μ' on F, for all random variables n_0, n_0' on F of law μ and μ' respectively, there is a coupling $((m_t^*(n_0))_{t\geq 0}, (m_t^*(n_0'))_{t\geq 0})$ of $(m_t(n_0))_{t\geq 0}, (m_t(n_0'))_{t\geq 0})$ such that

$$\mathbb{E}\|m_t^*(n_0) - m_t^*(n_0')\|_F \leq \mathtt{R}\gamma_{\mathrm{mix}}(t), \tag{1.23}$$

for all $t \geq 0$.

Let $\theta\colon F\to\mathbb{R}$ be continuous and bounded on bounded sets of F. A consequence of (1.19)-(1.23) is that, for $\alpha\geq 0$, the *resolvent*

$$R_\alpha\theta(n):=\int_0^\infty e^{-\alpha t}\mathbb{E}\theta(m_t(n))dt \tag{1.24}$$

is well defined, under the additional condition, in the limiting case $\alpha=0$, that θ is Lipschitz-continuous on bounded sets of F and satisfy the cancellation condition $\langle\theta,\nu\rangle=0$. Indeed, given such a function θ, denoting by $L_{\mathtt{R}}$ the Lipschitz constant of θ on the closed ball of center 0 and radius $\mathtt{R}$ in F, it follows from (1.23) that

$$|\mathbb{E}\theta(m_t(n))|=|\mathbb{E}\theta(m_t^*(n_0))-\mathbb{E}\theta(m_t^*(n_0'))|\leq L_{\mathtt{R}}\mathtt{R}\gamma_{\mathrm{mix}}(t),$$

where $n_0=n$ a.s. and n_0' follows the law ν. We will apply this result to the case where θ is a linear map $F\to\mathbb{R}^d$, and in particular to the case $\theta(n)=\chi(n)$, where

$$\chi(n)=K(1)\nabla_x n=\int_V vv_i\partial_{x_i}nd\nu(v)\in\mathbb{R}^d. \tag{1.25}$$

Then $R_\alpha\chi(n)$ is well defined in $C^2(\mathbb{T}^d)$ and satisfies the estimate

$$\|R_\alpha\chi(n)\|_{C^2(\mathbb{T}^d)}\leq\mathtt{R}\|\gamma_{\mathrm{mix}}\|_{L^1(\mathbb{R}_+)}, \tag{1.26}$$

for all $\alpha\geq 0$. Our last assumption is the following one. We consider a bounded linear functional Λ, with norm $\|\Lambda\|$, on $\left[C^2(\mathbb{T}^d)\right]^d$. By composition, we may consider the functional

$$\Lambda\circ R_0\circ\chi\colon n\mapsto\Lambda\left[(R_0\chi(n))\right] \tag{1.27}$$

on F and the action of the generator A on (1.27) and the square of (1.27). We will assume the following bounds regarding this action: there exists a constant $C^0_{\mathtt{R}}\geq 0$ such

$$|A\left|\Lambda\circ R_0\circ\chi\right|^2(n)|\leq C^0_{\mathtt{R}}\|\Lambda\|^2,\quad |A[\Lambda\circ R_0\circ\chi](n)|\leq C^0_{\mathtt{R}}\|\Lambda\|, \tag{1.28}$$

for all n with $\|n\|_F\leq\mathtt{R}$. We use (1.28) in the estimate (4.32) in particular.

2 Generator

2.1 Notations

The three first moments of a function $f\in L^1(V,\nu)$ are denoted by

$$\rho(f)=\int_V f(v)d\nu(v),\quad J(f)=\int_V vf(v)d\nu(v),\quad K(f)=\int_V v\otimes vf(v)d\nu(v). \tag{2.1}$$

We use the letter L to denote the linear operator

$$L\colon f\mapsto\rho(f)M-f \tag{2.2}$$

defined on $L^1(V,\nu)$. If E is a Banach space and I an interval in $\mathbb{R}$, we denote by $D(I;E)$ the Skorohod space of càdlàg functions from I to E (see [3], [16]). We denote by $\langle f,g\rangle$ the canonical scalar product on $L^2(\mathbb{T}^d\times V)$:

$$\langle f,g\rangle=\iint_{\mathbb{T}^d\times V}f(x,v)g(x,v)dxd\nu(v).$$

2.2 Resolution of the kinetic equation

We consider here the resolution of the Cauchy Problem (1.1)-(1.2) at fixed ε. We may therefore take $\varepsilon = 1$ for simplicity. We will solve (1.1)-(1.2) pathwise besides and, more exactly, we will construct a continuous solution map

$$((\bar{m}_t), f_{\rm in}) \mapsto f^\varepsilon.$$

Since only $q_t(x) := \nabla_x \bar{m}_t(x)$ does matter here, let us fix $T > 0$ and consider the equation

$$\partial_t f + v \cdot \nabla_x f = Lf + \rho(f) v \cdot q, \tag{2.3}$$

where $q\colon \mathbb{T}^d \times (0,T) \to \mathbb{R}^d$ is measurable and L is defined by (2.2). Let $\Phi_t(x,v) = (x+tv, v)$ denote the flow associated to the field $(v,0)$. Note that Φ_t preserves the measure on $\mathbb{T}^d \times V$.

Definition 2.1. Let $f_{\rm in} \in L^1(\mathbb{T}^d \times V)$, let $q\colon \mathbb{T}^d \times (0,T) \to \mathbb{R}^d$ be continuous with respect to x and càdlàg in t. A continuous function from $[0,T]$ to $L^1(\mathbb{T}^d \times V)$ is said to be a mild solution to (2.3) with initial datum $f_{\rm in}$ if

$$f(t) = e^{-t} f_{\rm in} \circ \Phi_{-t} + \int_0^t e^{-(t-s)} [\rho(f(s))(M + v \cdot q(s,\cdot))] \circ \Phi_{-(t-s)} ds, \tag{2.4}$$

for all $t \in [0,T]$.

Theorem 2.1. *Let $f_{\rm in} \in L^1(\mathbb{T}^d \times V)$, let $q\colon \mathbb{T}^d \times (0,T) \to \mathbb{R}^d$ be continuous with respect to x and càdlàg in t. There exists a unique mild solution to (2.3) in $C([0,T]; L^1(\mathbb{T}^d \times V))$ with initial datum $f_{\rm in}$. It satisfies*

$$\|f(t)\|_{L^1(\mathbb{T}^d \times V)} \leq e^{t\|q\|_{L^\infty(\mathbb{T}^d \times (0,t))}} \|f_{\rm in}\|_{L^1(\mathbb{T}^d \times V)} \quad \text{for all } t \in [0,T]. \tag{2.5}$$

In particular, we can write

$$f(t) = \Psi_t(f_{\rm in}, (q_s)_{0 \leq s \leq t}), \tag{2.6}$$

where Ψ_t is a continuous map from $L^1(\mathbb{T}^d \times V) \times D([0,t]; C(\mathbb{T}^d))$ to $L^1(\mathbb{T}^d \times V)$.

Proof of Theorem 2.1. Let E_T denote the space of continuous functions from $[0,T]$ to $L^1(\mathbb{T}^d \times \mathbb{R}^d)$. We use the norm

$$\|f\|_{E_T} = \sup_{t \in [0,T]} e^{-t\|q\|_{L^\infty(\mathbb{T}^d \times (0,T))}} \|f(t)\|_{L^1(\mathbb{T}^d \times V)}$$

on E_T. Note that

$$\|\rho(f)\|_{L^1(\mathbb{T}^d)} \leq \|f\|_{L^1(\mathbb{T}^d \times \mathbb{R}^d)}. \tag{2.7}$$

Let $f \in E_T$. Assume that (2.4) is satisfied. Then, by (2.7), and due to the fact that $v \in V$ has a norm less than 1, we have

$$\|f(t)\|_{L^1(\mathbb{T}^d \times V)} \leq e^{-t} \|f_{\rm in}\|_{L^1(\mathbb{T}^d \times V)} + (1 + \|q\|_{L^\infty(\mathbb{T}^d \times (0,T))}) \int_0^t e^{-(t-s)} \|f(s)\|_{L^1(\mathbb{T}^d \times V)} ds.$$

By Gronwall's Lemma applied to $t \mapsto e^t \|f(t)\|_{L^1(\mathbb{T}^d \times V)}$, we obtain (2.5) as an a priori estimate. Besides, the L^1-norm of the integral term in (2.4) can be estimated by $(e^{t\|q\|_{L^\infty(\mathbb{T}^d \times (0,T))}} - e^{-t})\|f\|_{E_T}$. This means that the application which, to $f \in E_t$, associates the right-hand side of (2.4), is a contraction (with factor $1 - e^{-T}$) of E_T. Therefore existence and uniqueness of a solution to (2.4) in $L^1(\Omega; E_T)$ follow from the Banach fixed point Theorem. By linearity of the equation, (2.6) follows from (2.5). ☐

To complete Theorem 2.1, we give the following result.

Proposition 2.2 (Non-negative solutions)**.** *Let $f_{\text{in}} \in L^1(\mathbb{T}^d \times V)$, let $q\colon \mathbb{T}^d \times (0,T) \to \mathbb{R}^d$ be continuous with respect to x and càdlàg in t. Let f be the unique mild solution to (2.3) in $C([0,T]; L^1(\mathbb{T}^d \times V))$ with initial datum f_{in}. Assume that*

$$M(v) + v \cdot q(t,x) \geq 0, \tag{2.8}$$

for a.e. $(t,x,v) \in (0,T) \times \mathbb{T}^d \times V$ and that $f_{\text{in}} \geq 0$ a.e. Then $f \geq 0$ a.e. on $(0,T) \times \mathbb{T}^d \times V$.

Proof of Proposition 2.2. In view of (2.4), it is sufficient to show that $\rho(f) \geq 0$ a.e. on $(0,T) \times \mathbb{T}^d$. We have

$$\rho(f)(t) = e^{-t}\rho(f_{\text{in}} \circ \Phi_{-t}) + (1-e^{-t}) \int_0^t \int_V \frac{e^{-(t-s)}}{1-e^{-t}} [\rho(f(s))(M + v \cdot q(s,\cdot))] \circ \Phi_{-(t-s)} ds d\nu.$$

By convexity of $s \mapsto s^-$, we deduce that

$$\begin{aligned}[\rho(f)(t)]^- &\leq e^{-t}[\rho(f_{\text{in}} \circ \Phi_{-t})]^- \\ &\quad + (1-e^{-t}) \int_0^t \int_V \frac{e^{-(t-s)}}{1-e^{-t}} [\rho(f(s))(M + v \cdot q(s,\cdot))]^- \circ \Phi_{-(t-s)} ds d\nu.\end{aligned}$$

Using (2.8) and $f_{\text{in}} \geq 0$, we obtain

$$[\rho(f)(t)]^- \leq \int_0^t \int_V e^{-(t-s)} [\rho(f(s))^- (M + v \cdot q(s,\cdot))] \circ \Phi_{-(t-s)} ds d\nu \tag{2.9}$$

We integrate (2.9) over $x \in \mathbb{T}^d$. Since Φ_t is measure preserving, we obtain

$$e^t \|\rho(f)^-\|_{L^1(\mathbb{T}^d)}(t) \leq \int_0^t e^s \|\rho(f)^-\|_{L^1(\mathbb{T}^d)}(s) ds.$$

By the Gronwall Lemma, we deduce that $\rho(f)^- = 0$ a.e. on $(0,T) \times \mathbb{T}^d$. ☐

We will also need the following result about the regularity of solutions.

Proposition 2.3 (Propagation of regularity)**.** *Let $f_{\text{in}} \in L^1(\mathbb{T}^d \times V)$ satisfy*

$$f_{\text{in}} \in L^2(\mathbb{T}^d \times V), \quad \partial_{x_i} f_{\text{in}} \in L^2(\mathbb{T}^d \times V), \tag{2.10}$$

for $i \in \{1, \dots, d\}$. Let $q\colon \mathbb{T}^d \times (0,T) \to \mathbb{R}^d$ be of class C^1 with respect to x, with $q(x,t)$, $\partial_{x_i} q(x,t)$ càdlàg in t for every x. Let f be the unique mild solution to (2.3) in $C([0,T]; L^1(\mathbb{T}^d \times V))$ with initial datum f_{in}. Assume (1.5) and let

$$D = \left[\alpha^{-2} + \|q\|^2_{L^\infty(\mathbb{T}^d \times (0,T))} + \|\nabla_x q\|^2_{L^\infty(\mathbb{T}^d \times (0,T))}\right]^{1/2}.$$

Then $f(t)$ and $\nabla_x f(t) \in L^2(\mathbb{T}^d \times V)$ for all $t \in [0,T]$, and we have the estimate

$$\|f(t)\|^2_{L^2(\mathbb{T}^d \times V)} + \sum_{i=1}^d \|f(t)\|^2_{L^2(\mathbb{T}^d \times V)} \leq e^{Dt} \|f_{\text{in}}\|_{L^2(\mathbb{T}^d \times V)}, \tag{2.11}$$

for all $t \in [0,T]$

Proof of Proposition 2.3. The mild solution f to (2.3) is obtained by a fixed-point argument. Therefore f is the limit, in $C([0,T];L^1(\mathbb{T}^d\times V))$, of the iterative sequence f^k defined by $f^0=f_{\rm in}$, f^{k+1} solution to the equation

$$\partial_t f^{k+1}+v\cdot\nabla_x f^{k+1}=\rho(f^k)[M(v)+v\cdot q]-f^{k+1}, \tag{2.12}$$

with initial condition $f_{\rm in}$, in the sense that

$$f^{k+1}(t)=e^{-t}f_{\rm in}\circ\Phi_{-t}+\int_0^t e^{-(t-s)}[\rho(f^k(s))(M+v\cdot q(s,\cdot))]\circ\Phi_{-(t-s)}ds, \tag{2.13}$$

for all $t\in[0,T]$. On the basis of (2.13), using (1.5), we derive the L^2-bound

$$\|f^{k+1}(t)\|_{L^2(\mathbb{T}^d\times V)}\leq e^{-t}\|f_{\rm in}\|_{L^2(\mathbb{T}^d\times V)}+C\int_0^t e^{-(t-s)}\|\rho(f^k(s))\|_{L^2(\mathbb{T}^d)}ds.$$

with

$$C=\left[\alpha^{-2}+\|q\|^2_{L^\infty(\mathbb{T}^d\times(0,T))}\right]^{1/2}.$$

Since $\|\rho(f)\|_{L^2(\mathbb{T}^d)}\leq\|f\|_{L^2(\mathbb{T}^d\times V)}$ by Jensen's Inequality, we obtain

$$\varphi_{k+1}(t)\leq\|f_{\rm in}\|_{L^2(\mathbb{T}^d\times V)}+C\int_0^t\varphi_k(s)ds,\quad \varphi_k(t)=e^t\|f^k(t)\|_{L^2(\mathbb{T}^d\times V)}.$$

We conclude that

$$\|f^k(t)\|_{L^2(\mathbb{T}^d\times V)}\leq e^{Ct}\|f_{\rm in}\|_{L^2(\mathbb{T}^d\times V)},$$

which gives a similar bound for f at the limit $k\to+\infty$. The bound on the derivatives with respect to x of f is obtained similarly on the basis of (2.13), by differentiation and L^2-estimate as above. We conclude to (2.11). □

2.3 Generator

We emphasize the property (2.6) in Theorem 2.1 because it is used in the proof of the following Markov property. We will not give the details of the proof however; we refer to Theorem 4.3 in [10] instead.

Theorem 2.4 (Markov property). *Let $(\bar m_t)$ be an admissible pilot process in the sense of Definition 1.1. Let $\mathcal{X}$ denote the state space*

$$\mathcal{X}=L^1(\mathbb{T}^d\times V)\times F. \tag{2.14}$$

For $(f,n)\in\mathcal{X}$, let f_t denote the mild solution to (2.3) with initial datum f and forcing $F_t=\nabla_x m_t(n)$. Then $(f_t,m_t(n))_{t\geq 0}$ is a time-homogeneous Markov process over $\mathcal{X}$.

We denote by (P_t) the semi-group associated to $(f_t,m_t(n))_{t\geq0}$:

$$P_t\varphi(f,n):=\mathbb{E}_{(f,n)}\varphi(f_t,m_t),\quad \varphi\in C_b(\mathcal{X}). \tag{2.15}$$

Coming back to the case $\varepsilon>0$ (instead of $\varepsilon=1$), we obtain that, denoting by f^ε_t the mild solution to (1.1), the process $(f^\varepsilon_t,\bar m_{\varepsilon^{-2}t})$ is a Markov. Formally, its generator $\mathscr{L}^\varepsilon$ is given as

$$\mathscr{L}^\varepsilon=\frac{1}{\varepsilon^2}\mathscr{L}_\sharp+\frac{1}{\varepsilon}\mathscr{L}_\flat,$$

where $\mathscr{L}_\sharp$ and $\mathscr{L}_\flat$ are defined by

$$\mathscr{L}_\sharp\varphi(f,n) = A\varphi(f,n) + (Lf + \rho(f)v\cdot\nabla_x n, D_f\varphi(f,n)), \tag{2.16}$$
$$\mathscr{L}_\flat\varphi(f,n) = -(v\cdot\nabla_x f, D_f\varphi(f,n)). \tag{2.17}$$

We will not characterize $\mathscr{L}^\varepsilon$ (it is very difficult to identify its domain), but simply prove the following result, which describes some functions in the domains of $\mathscr{L}_\sharp$ and $\mathscr{L}_\flat$.

Proposition 2.5. *Let $(\bar{m}_t)$ be an admissible pilot process in the sense of Definition 1.1. Let A be the generator of $(\bar{m}_t)$, let $\mathcal{X}$ be the state space defined by (2.14), and let $\mathscr{L}_\sharp$ and $\mathscr{L}_\flat$ be defined by (2.16)-(2.17). Let $\xi \in C^1(\mathbb{T}^d)$, $\theta \in \mathcal{D}(A)$ and $Q(v)$ be a polynomial in v. The function*

$$\gamma\colon (f,n) \mapsto \iint_{\mathbb{T}^d\times V} f(x,v)\xi(x)\theta(n)(x)Q(v)dxd\nu(v) \tag{2.18}$$

defined on $\mathcal{X}$ satisfies $\mathscr{L}_\sharp\gamma(f,n), \mathscr{L}_\flat\gamma(f,n) < +\infty$ for all $(f,n)\in\mathcal{X}$ and γ is in the domain of $\mathscr{L}^\varepsilon$ in the sense that

$$P_t^\varepsilon\gamma(f,n) = \gamma(f,n) + t\mathscr{L}^\varepsilon\gamma(f,n) + o(t), \tag{2.19}$$

for all $(f,n)\in\mathcal{X}$, where P_t is defined by (2.15) and (P_t^ε) is the rescaled semi-group $P_t^\varepsilon = P_{\varepsilon^{-2}t}$.

Corollary 2.6. *Any differentiable combination φ of functions of the form (2.18) or n satisfies $\mathscr{L}_\sharp\varphi(f,n), \mathscr{L}_\flat\varphi(f,n) < +\infty$ for all $(f,n)\in\mathcal{X}$ and φ is in the domain of $\mathscr{L}^\varepsilon$ in the sense that*

$$P_t^\varepsilon\varphi(f,n) = \varphi(f,n) + t\mathscr{L}^\varepsilon\varphi(f,n) + o(t), \tag{2.20}$$

for all $(f,n)\in\mathcal{X}$

We will note give the proof of Proposition 2.5, which is elementary, nor give the proof of Corollary 2.6, but we need to specify what we mean precisely by "differentiable combination". The function φ in Corollary 2.6 is of the form

$$\varphi\colon (f,n) \mapsto \psi(\gamma_1(f,n),\dots,\gamma_m(f,n);n), \tag{2.21}$$

where each function γ_i is as (2.18) and the function $\psi\colon \mathbb{R}^m\times F\to\mathbb{R}$ has the following properties:

1. for all $u\in\mathbb{R}^m$, $n\mapsto\psi(u;n)$ is in the domain of A and $(u,n)\mapsto A\psi(u;n)$ is bounded on bounded sets of $\mathbb{R}^m\times F$,
2. for all $n\in F$, $u\mapsto\psi(u;n)$ is differentiable, $(u,n)\mapsto\nabla_u\psi(u;n)$ is bounded on bounded sets of $\mathbb{R}^m\times F$ and continuous with respect to n.

Note then that φ^2 is also of the form (2.21). Corollary 2.6 has therefore the following consequence.

Corollary 2.7 (Martingale)**.** *Let φ be given by (2.21). Let $f_{\mathrm{in}}^\varepsilon\in L^1(\mathbb{T}^d\times V)$ and let f_t^ε be the mild solution to (1.1) in $C([0,T];L^1(\mathbb{T}^d\times V))$ with initial datum $f_{\mathrm{in}}^\varepsilon$. Then, for all $n\in F$,*

$$M_\varphi^\varepsilon(t) := \varphi(f_t^\varepsilon, m_t^\varepsilon(n)) - \varphi(f_{\mathrm{in}}^\varepsilon, n) - \int_0^t \mathscr{L}^\varepsilon\varphi(f_s^\varepsilon, m_s^\varepsilon(n))ds \tag{2.22}$$

is a martingale with quadratic variation

$$\langle M_\varphi^\varepsilon, M_\varphi^\varepsilon\rangle_t = \int_0^t \left[\mathscr{L}^\varepsilon\varphi^2(f_s^\varepsilon, m_s^\varepsilon(n)) - 2\varphi(f_s^\varepsilon, m_s^\varepsilon(n))\mathscr{L}^\varepsilon\varphi(f_s^\varepsilon, m_s^\varepsilon(n))\right]ds, \tag{2.23}$$

for all $t\in[0,T]$.

Proof of Corollary 2.7. We refer to Appendix B in [10]. ☐

2.4 Main generator and invariant solutions

The main generator $\mathscr{L}_\sharp$, defined in (2.16), is associated to the following equation (started at $t = t_0$)

$$\begin{cases} \frac{d}{dt} g_t = L g_t + \rho(g_t) v \cdot \nabla_x m(t, t_0; n) \\ g_{t_0} = g. \end{cases} \tag{2.24}$$

A solution to (2.24) satisfies (formally in a first step) $\rho(g_t) = \rho(g)$. Therefore (2.24) is a simple dissipative equation on g_t, with source term. The explicit solution to (2.24) reads

$$g_{t_0,t} = e^{-(t-t_0)} g + \int_{t_0}^{t} e^{-(t-s)} \left(\rho(g) M + \rho(g) v \cdot \nabla_x m_{t_0,s}\right) ds, \tag{2.25}$$

i.e.

$$g_{t_0,t} = e^{-(t-t_0)} g + \rho(g) M \left(1 - e^{-(t-t_0)}\right) + \rho(g) v \nabla_x \int_{t_0}^{t} m_{t_0,s} e^{-(t-s)} ds. \tag{2.26}$$

We prove the following result, in time $t_0 \to -\infty$.

Proposition 2.8. *Let $(\bar{m}_t)$ be an admissible pilot process in the sense of Definition 1.1. Let $g \in L^1(V)$ and let $g_{t_0,t}$ be defined by (2.26). Define*

$$\overline{g}_t = \rho(g) M + \rho(g) v \cdot \nabla_x \bar{w}_t, \quad \bar{w}_t := \int_{-\infty}^{t} e^{-(t-s)} \bar{m}_s ds. \tag{2.27}$$

Then, for fixed $t \in \mathbb{R}$, $n \in F$, the couple $(g_{t_0,t}, m(t, t_0; n))$ is converging in law when $t_0 \to -\infty$ to $(\overline{g}_t, \bar{m}_t)$.

Proof of Proposition 2.8. Our aim is to prove that

$$\lim_{t_0 \to +\infty} \mathbb{E}\Phi(g_{t_0,t}, m(t, t_0; n)) = \mathbb{E}\Phi(\overline{g}_t, \bar{m}_t), \tag{2.28}$$

for all continuous and bounded $\Phi \colon L^1(V) \times F \to \mathbb{R}$. Actually, it is sufficient to consider uniformly continuous and bounded functions Φ (*cf.* Proposition I-2.4 in [15]). For a given $\varepsilon > 0$, let us fix therefore a modulus of uniform continuity η of Φ associated to ε. Let us apply the mixing hypothesis (1.23) with $n_0 = n$ a.s., n_0' of law the invariant measure λ. Then $(m(t, t_0; n), \bar{m}_t)_{t \geq t_0}$ has the same law as $(m^*_{t-t_0}(n_0), m^*_{t-t_0}(n_0'))_{t \geq t_0}$. It follows that

$$\mathbb{E}\Phi(g_{t_0,t}, m(t, t_0; n)) - \mathbb{E}\Phi(\overline{g}_t, \bar{m}_t) = \mathbb{E}\Phi(g^*_{t_0,t}, m^*_{t_0,t}) - \mathbb{E}\Phi(\overline{g}^*_t, \bar{m}^*_{t_0,t}), \tag{2.29}$$

where, we have denoted $m^*_{t_0,t} = m^*_{t-t_0}(n_0)$ and $\bar{m}^*_{t_0,t} = m^*_{t-t_0}(n_0')$ for simplicity, and have set

$$g^*_{t_0,t} = e^{-(t-t_0)} g + \rho(g) M \left(1 - e^{-(t-t_0)}\right) + \rho(g) v \cdot \nabla_x \int_{t_0}^{t} m^*_{t_0,s} e^{-(t-s)} ds,$$

and

$$\overline{g}_t = \rho(g) M + \rho(g) v \cdot \nabla_x \int_{-\infty}^{t} e^{-(t-s)} \bar{m}^*_{t_0,s} ds.$$

By (2.29), we have

$$\begin{aligned} &|\mathbb{E}\Phi(g_{t_0,t}, m(t, t_0; n)) - \mathbb{E}\Phi(\overline{g}_t, \bar{m}_t)| \\ &\qquad \leq 2\|\Phi\|_{C_b} \varepsilon + \mathbb{P}(\|g^*_{t_0,t} - \overline{g}^*_t\|_{L^1(V)} > \eta) + \mathbb{P}(\|m^*_{t_0,t} - \bar{m}^*_{t_0,t}\|_F > \eta). \end{aligned} \tag{2.30}$$

By (1.23) and the Markov inequality, we have

$$\mathbb{P}(\|m^*_{t_0,t} - \bar{m}^*_{t_0,t}\|_F > \eta) \leq \frac{1}{\eta}\mathbb{E}\|m^*_{t_0,t} - \bar{m}^*_{t_0,t}\|_F \leq \frac{\mathtt{R}}{\eta}\gamma_{\mathrm{mix}}(t-t_0). \tag{2.31}$$

The mixing hypothesis (1.23) also gives the estimate

$$\begin{aligned}\mathbb{E}\|g^*_{t_0,t} - \overline{g}^*_t\|_{L^1(V)} \leq 2\rho(|g|)e^{-(t-t_0)} + \rho(|g|)\mathtt{R}\int_{t_0}^t e^{-(t-s)}\gamma_{\mathrm{mix}}(s-t_0)ds \\ + \rho(|g|)\int_{-\infty}^{t_0} e^{-(t-s)}\|\bar{m}^*_{t_0,s}\|_F ds.\end{aligned}$$

Using the bound (1.22), we obtain

$$\mathbb{E}\|g^*_{t_0,t} - \overline{g}^*_t\|_{L^1(V)} \leq (2+\mathtt{R})\rho(|g|)e^{-(t-t_0)} + \rho(|g|)\mathtt{R}\int_0^{t-t_0} e^{-(t-s-t_0)}\gamma_{\mathrm{mix}}(s)ds. \tag{2.32}$$

By the Markov inequality and the fact that γ_{mix} is integrable, we conclude that

$$\mathbb{P}(\|g^*_{t_0,t} - \overline{g}^*_t\|_{L^1(V)} > \eta) \leq \eta^{-1}\omega(t_0),$$

where the quantity $\omega(t_0)$, which depends on t, g, $\mathtt{R}$, tends to 0 when $t_0 \to +\infty$. Combining this estimate with (2.30) and (2.31), we get the desired result. □

For each fixed $\rho \in \mathbb{R}$, (2.24) defines a stochastic dynamical system in $L^1_\rho(V) \times F$, where

$$L^1_\rho(V) = \left\{g \in L^1(V); \rho(g) = \rho\right\}.$$

Proposition 2.8 shows that (2.24) has a unique, ergodic invariant measure μ_ρ on $L^1_\rho(V) \times F$ defined by

$$\langle \mu_\rho, \Phi\rangle = \mathbb{E}\Phi(\rho\bar{M}_t, \bar{m}_t), \tag{2.33}$$

where

$$\bar{M}_t = M + v\cdot\nabla_x\bar{w}_t, \tag{2.34}$$

(the process $\bar{w}_t$ is defined in (2.27)). By ergodicity hence, a continuous and bounded function Φ on $L^1(V)\times F$ satisfies $\mathscr{L}_\sharp\Phi = 0$ if, and only if, Φ is constant. Later on, we will consider the action of $\mathscr{L}_\sharp$ on continuous and bounded function Φ on $L^1(\mathbb{T}^d \times V)\times F$. In that case, the space variable x is simply a parameter. We will have $\mathscr{L}_\sharp\Phi = 0$ then, if, and only if, Φ is a function of $\rho(g)$: $\Phi(g,n) = \bar{\Phi}(\rho(g))$. In that case, $\langle\Phi, \mu_\rho\rangle = \bar{\Phi}(\rho)$. An other consequence of the ergodic character of μ_ρ is that cancellation against μ_ρ is a sufficient and necessary condition for the Poisson equation $\mathscr{L}_\sharp\Psi = \Phi$ to be solvable, see Proposition 2.9 below. Before we state this proposition, let us introduce $P^\sharp_t$, the semi-group associated to $\mathscr{L}_\sharp$: it is defined, for $(g,n) \in \mathcal{X} = L^1(\mathbb{T}^d \times V)\times F$ and Φ a continuous and bounded function on $\mathcal{X}$, by the formula

$$P^\sharp_t\Phi(g,n) = \mathbb{E}\Phi(g_{0,t}, m_t(n)) \tag{2.35}$$

where $g_{t_0,t}$ is defined by (2.26). Note that the trajectory $t\mapsto (g_{0,t}, m_t(n))$ remains in a bounded set of $\mathcal{X}$, a.s. (*cf.* the proof of Proposition 2.9 and Lemma 2.10 for a precise bound). Consequently, the formula (2.35) can be extended to functions Φ which are continuous and bounded on bounded sets of $\mathcal{X}$.

Proposition 2.9 (Poisson equation). *Let $(\bar m_t)$ be an admissible pilot process in the sense of Definition 1.1. Let $\mathcal{X}$ be the state space defined by (2.14). Let $\mathscr{L}_\sharp$ be defined by (2.16) and let μ_ρ be defined by (2.33). Let $\Phi\colon\mathcal{X}\to\mathbb{R}$ be Lipschitz continuous on bounded sets of $\mathcal{X}$. Assume $\langle\Phi,\mu_\rho\rangle=0$ and let*

$$\Psi(g,n)=-\int_0^\infty P_t^\sharp\Phi(g,n)dt. \tag{2.36}$$

Then Ψ is solution to the Poisson equation $\mathscr{L}_\sharp\Psi=\Phi$ in the sense that

$$\lim_{t\to0+}\frac{P_t^\sharp\Psi(g,n)-\Psi(g,n)}{t}=\Phi(g,n), \tag{2.37}$$

for all $(g,n)\in\mathcal{X}$.

Lemma 2.10 (Poisson equation, bounds). *Under the hypothesis of Proposition 2.9, let $K_{g,n}$ be the set of $(f,m)\in\mathcal{X}$ such that $\|f\|_{L^1(\mathbb{T}^d\times V)}\leq\|g\|_{L^1(\mathbb{T}^d\times V)}(2+\mathtt{R})$ and $\|m\|_F\leq\mathtt{R}$. Let $C_\Phi(\|g\|_{L^1(\mathbb{T}^d\times V)},\mathtt{R})$ be the Lipschitz constant of Φ on $K_{g,n}$. Then Ψ defined by (2.36) satisfies the bound*

$$|\Psi(g,n)|\leq C_\Phi(\|g\|_{L^1(\mathbb{T}^d\times V)},\mathtt{R})\Big[(2+\mathtt{R}(1+\|\gamma_{\mathrm{mix}}\|_{L^1(\mathbb{R}_+)}))\|g\|_{L^1(\mathbb{T}^d\times V)}+\mathtt{R}\|\gamma_{\mathrm{mix}}\|_{L^1(\mathbb{R}_+)}\Big], \tag{2.38}$$

for all $g\in L^1(\mathbb{T}^d\times V)$ and for all n with $\|n\|_F\leq\mathtt{R}$.

Proof of Proposition 2.9 and Lemma 2.10. We use the same coupling as in the proof of Proposition 2.8:

$$\mathbb{E}\Phi(g_{0,t},m_t(n))=\mathbb{E}\Phi(g_{0,t},m_t(n))-\langle\Phi,\mu_\rho\rangle=\mathbb{E}\Phi(g^*_{0,t},m^*(t;n))-\mathbb{E}\Phi(\overline{g}^*_t,\bar m^*_t).$$

By (2.26), (2.27), $g^*_{0,t}$ and $\overline{g}^*_t$ are in the closed ball of center 0 and radius $\|g\|_{L^1(\mathbb{T}^d\times V)}(2+\mathtt{R})$ of $L^1(\mathbb{T}^d\times V)$. Consequently, we have

$$|\mathbb{E}\Phi(g_{0,t},m_t(n))|\leq C_\Phi(\|g\|_{L^1(\mathbb{T}^d\times V)},\mathtt{R})\left[\mathbb{E}\|g^*_{0,t}-\overline{g}^*_t\|_{L^1(\mathbb{T}^d\times V)}+\mathbb{E}\|m^*(t;n)-\bar m^*_t\|_F\right].$$

By (1.23) and (2.32) (it is trivial to generalize the latter estimate to x-dependent functions g), we deduce that

$$\begin{aligned}|\mathbb{E}\Phi(g_{0,t},m_t(n))|\leq C_\Phi(\|g\|_{L^1(\mathbb{T}^d\times V)},\mathtt{R})\Big[&(2+\mathtt{R})\|g\|_{L^1(\mathbb{T}^d\times V)}e^{-t}\\&+\|g\|_{L^1(\mathbb{T}^d\times V)}\mathtt{R}\int_0^t e^{-(t-s)}\gamma_{\mathrm{mix}}(s)ds+\mathtt{R}\gamma_{\mathrm{mix}}(t)\Big].\end{aligned}$$

Consequently, (2.36) makes sense and Ψ satisfies the bound (2.38). It is easy to obtain (2.37) then, *cf.* the proof of Proposition 3.4 in [21]. □

We will also need the following result.

Lemma 2.11 (Poisson equation, differentiation). *Under the hypotheses of Proposition 2.9, assume that, for all $n\in F$, $(g,n)\mapsto\Phi(g,n)$ is differentiable with respect to $g\in L^1(V)$ and that $(g,n)\mapsto D_g\Phi(g,n)$ is bounded on bounded sets. Let Ψ be defined by (2.36). Then, for all $n\in F$, $(g,n)\mapsto\Psi(g,n)$ is also differentiable with respect to $g\in L^1(V)$ and*

$$D_g\Psi(g,n)\cdot h=-\int_0^\infty e^{-t}\mathbb{E}D_g\Phi(g_{0,t},m_t(n))\cdot h dt.$$

In particular, we have the bound

$$|D_g\Psi(g,n)\cdot h|\leq \sup_{(f,m)\in K_{g,n}}\|D_g\Phi(f,m)\|_{L^1(V)\to L^1(V)}\|h\|_{L^1(V)},$$

where $K_{g,n}$ is the bounded set defined in Lemma 2.10.

Proof of Lemma 2.11. The result follows from the fact that $S_t\colon g\mapsto g_{0,t}$, with $g_{0,t}$ equal to (2.26) is affine ($S_t g=e^{-t}g+\cdots$). ☐

3 The perturbed test-function method

To prove the convergence of $(\rho^\varepsilon)_\varepsilon$, we use the perturbed test function method, [19]. The limit generator $\mathscr{L}$ acts on functions φ of the variable $\rho\in L^1(\mathbb{T}^d)$. They can be seen as functions on the state-space $\mathcal{X}$ (see (2.14)) simply by considering the map

$$(f,n)\mapsto\varphi(\rho(f)),$$

which we still denote by φ. We perturb this initial test function into

$$\varphi^\varepsilon=\varphi+\varepsilon\varphi^1+\varepsilon^2\varphi^2, \tag{3.1}$$

in order to control $\mathscr{L}^\varepsilon\varphi^\varepsilon$ as follows:

$$\mathscr{L}^\varepsilon\varphi^\varepsilon=\mathscr{L}\varphi+o(1). \tag{3.2}$$

This gives the identification of the limit generator $\mathscr{L}$. A precise estimate of the form (3.2) is given in Section 3.1.3 for functions φ of the form

$$\varphi(\rho)=\psi(\langle\rho,\xi\rangle), \tag{3.3}$$

where $\xi\in C^\infty(\mathbb{T}^d)$ and ψ is a Lipschitz function on $\mathbb{R}$ such that $\psi'\in C_b^\infty(\mathbb{R})$. It is sufficient to consider such test functions since they altogether form a separating class (indeed, they form a set that separates points, and, by Theorem 4.5 p. 113 in [12], a separating class).

Then, in Section 4, we show that (ρ^ε) is tight and converges in law to the solution of the martingale problem associated to the limit generator $\mathscr{L}$. In the last Section 5, we show that this limit is the law of the solution to the stochastic PDE (1.11).

3.1 First and second correctors

3.1.1 First corrector

Let φ be given by (3.3). We introduce the development (3.1) in the asymptotic formula (3.2) and identify, formally, the powers of ε. At the order ε^{-2}, we obtain $\mathscr{L}_\sharp\varphi=0$. This identity is satisfied since φ is independent on n and the right-hand side of (1.1) is an element of the kernel of ρ:

$$\mathscr{L}_\sharp\varphi(f,n)=(\rho[Lf+\rho(f)v\cdot\nabla_x n],D_\rho\varphi(\rho(f)))=0.$$

At the order ε^{-1}, we obtain the equation for the first corrector:

$$\mathscr{L}_\sharp\varphi_1+\mathscr{L}_\flat\varphi=0. \tag{3.4}$$

We compute

$$\mathscr{L}_\flat\varphi(f,n) = -(\rho(v\cdot\nabla_x f), D_\rho\varphi(\rho(f))) = -(\mathrm{div}_x(J(f)), D_\rho\varphi(\rho(f))).$$

More exactly, in view of (3.3), we have

$$\mathscr{L}_\flat\varphi(f,n) = \psi'\big(\langle\rho(f),\xi\rangle\big)\langle J(f),\nabla_x\xi\rangle.$$

It is clear that we can apply Proposition 2.9 to $\Phi = \mathscr{L}_\flat\varphi$. We must simply check the cancellation condition

$$\langle\mu_\rho,\mathscr{L}_\flat\varphi\rangle = 0. \tag{3.5}$$

We examine (2.33)-(2.34): we have $J(\bar{M}_t) = K\nabla_x\bar{w}_t$ by (1.8), hence

$$\langle\mu_\rho,\mathscr{L}_\flat\varphi\rangle = -\mathbb{E}(\mathrm{div}_x(K\nabla_x\bar{w}_t), D_\rho\varphi(\rho)).$$

This gives (3.5) since $\mathbb{E}\bar{w}_t = 0$, due to the fact that $\bar{m}_t$ is centred (see (1.21)). By Proposition 2.9, we have the resolvent formula

$$\varphi_1(g,n) = \int_0^\infty \mathbb{E}\mathscr{L}_\flat\varphi(g_{0,t}, m_t(n))dt, \tag{3.6}$$

with $g_{t_0,t}$ given in (2.26). We compute, using the cancellation condition $J(M) = 0$ (*cf.* (1.8)),

$$J(g_{0,t}) = e^{-t}J(g) + \rho(g)K\nabla_x\int_0^t m_s(n)e^{-(t-s)}ds. \tag{3.7}$$

By explicit integration, it follows that

$$\varphi_1(f,n) = -(\mathrm{div}_x(J(f) + \rho(f)R_0\chi(n)), D_\rho\varphi(\rho(f))), \tag{3.8}$$

where $\chi(n)$ is defined in (1.25). More precisely, we have

$$\varphi_1(f,n) = \psi'\big(\langle\rho(f),\xi\rangle\big)\langle J(f) + \rho(f)R_0\chi(n),\nabla_x\xi\rangle. \tag{3.9}$$

Due to Lemma 2.10, φ_1 satisfies the bound (2.38) with a constant

$$C_\Phi(\theta,\mathtt{R}) = (\|\psi''\|_{L^\infty}\|\xi\|^2_{W^{1,\infty}(\mathbb{T}^d)} + \|\psi'\|_{L^\infty}\|\xi\|_{W^{1,\infty}(\mathbb{T}^d)})(1+\theta^2(2+\mathtt{R})^2).$$

In particular, we have, for all $f\in L^1(\mathbb{T}^d\times V)$ and $n\in F$ such that $\|n\|_F\leq\mathtt{R}$,

$$|\varphi_1(f,n)| \leq \bar{\Phi}_1(\|\psi'\|_{C^1_b(\mathbb{R})}, \|\xi\|_{C^1(\mathbb{T}^d)}, \|f\|_{L^1(\mathbb{T}^d\times V)}, \mathtt{R}), \tag{3.10}$$

where $\bar{\Phi}_1$ is an increasing function of its arguments and is bounded on bounded sets of $\mathbb{R}^4$.

3.1.2 Second corrector and limit generator

At order 1, the equation given by (3.1)-(3.2) is

$$\mathscr{L}_\sharp\varphi_2 + \mathscr{L}_\flat\varphi_1 = \mathscr{L}\varphi. \tag{3.11}$$

Due to Proposition 2.9, a necessary condition to solve (3.11) is that

$$\mathscr{L}\varphi(\rho) = \langle\mathscr{L}_\flat\varphi_1,\mu_\rho\rangle. \tag{3.12}$$

Equation (3.12) defines the limit generator $\mathscr{L}$. On the basis of (3.9), we compute

$$\begin{aligned}\mathscr{L}_\flat\varphi_1(f,n) = \psi''\big(\langle\rho(f),\xi\rangle\big)\langle J(f),\nabla_x\xi\rangle\langle J(f)+\rho(f)R_0\chi(n),\nabla_x\xi\rangle \\ + \psi'\big(\langle\rho(f),\xi\rangle\big)\left[\langle K(f)+J(f)\otimes R_0\chi(n),D_x^2\xi\rangle + \langle J(f),\nabla_x R_0\chi(n)\nabla_x\xi\rangle\right],\end{aligned} \tag{3.13}$$

where $\nabla_x R_0\chi(n)$ is the matrix with ij-entry $\partial_{x_i}R_0\chi^j(n)$. In a more formal way, starting from (3.8), the identity (3.13) can be expressed as

$$\begin{aligned}\mathscr{L}_\flat\varphi_1(f,n) = (D_x^2\colon K(f) + \mathrm{div}_x(\mathrm{div}_x(J(f))R_0\chi(n)), D_\rho\varphi(\rho(f))) \\ + D_\rho^2\varphi(\rho(f))\cdot(\mathrm{div}_x(J(f)+\rho(f)R_0\chi(n)),\mathrm{div}_x\, J(f)).\end{aligned} \tag{3.14}$$

In (3.14), the first-order and second-order terms (regarding differentiation with respect to ρ of φ) are more clearly identified than in (3.13). Let $g_{t_0,t}$ be given by (2.26). The first moment of $g_{0,t}$ has been evaluated in (3.7). A similar computation, using the cancellation of the third moment in (1.8), gives the expression of the second moment:

$$K(g_{0,t}) = e^{-t}K(g) + \rho(g)(1-e^{-t})K(M). \tag{3.15}$$

We obtain

$$\begin{aligned}\mathscr{L}\varphi(\rho) = (K(M)\colon D_x^2\rho + \mathbb{E}\,\mathrm{div}_x[\mathrm{div}_x[\rho\chi(\bar m_0)](R_1R_0\chi)(\bar m_0)], D_\rho\varphi(\rho)) \\ + \mathbb{E}D_\rho^2\varphi(\rho)\cdot(\mathrm{div}_x[\rho(R_1\chi+R_1R_0\chi)(\bar m_0)],\mathrm{div}_x[\rho\chi(\bar m_0)]).\end{aligned} \tag{3.16}$$

We have used the identity

$$J(\bar g_0) = \rho(g)K(1)\nabla_x\int_{-\infty}^0 e^t\bar m_t dt = \rho(g)\int_{-\infty}^0 e^t\chi(\bar m_t)dt,$$

and then, due to the fact that $(\bar m_t)$ is stationary, the following formula, valid for any continuous function $\Theta\colon F\to\mathbb{R}$,

$$\mathbb{E}[J(\bar g_0)\Theta(\bar m_0)] = \rho(g)\mathbb{E}\left[\chi(\bar m_0)(R_1\Theta)(\bar m_0)\right], \tag{3.17}$$

To treat the product $(\mathrm{div}_x(J(f)),\mathrm{div}_x(J(f)))$ in the second-order term of (3.14), we have used also the identity:

$$\begin{aligned}&\int_{-\infty}^0\int_{-\infty}^0 e^{t+s}\mathbb{E}D_\rho^2\varphi(\rho)\cdot(\rho\chi(\bar m_t),\rho\chi(\bar m_s))dsdt \qquad (3.18)\\ &=\int_0^\infty e^{-\sigma}\mathbb{E}D_\rho^2\varphi(\rho)\cdot(\rho\chi(\bar m_0),\rho\chi(\bar m_\sigma))d\sigma = \mathbb{E}D_\rho^2\varphi(\rho)\cdot(\rho\chi(\bar m_0),\rho(R_1\chi)(\bar m_0)),\end{aligned}$$

which follows, using the fact that $(\bar m_t)$ is stationary, from the change of variable $t=s+\sigma$ in (3.18). The complete proofs of (3.17) and (3.18) are given in Section 3.2 below. Next, we use the resolvent formula $R_0R_1 = R_1R_0 = R_0 - R_1$ to simplify (3.16) a bit:

$$\begin{aligned}\mathscr{L}\varphi(\rho) = (K(M)\colon D_x^2\rho + \mathbb{E}\,\mathrm{div}_x[\mathrm{div}_x[\rho\chi(\bar m_0)](R_0R_1\chi)(\bar m_0)], D_\rho\varphi(\rho)) \\ + \mathbb{E}D_\rho^2\varphi(\rho)\cdot(\mathrm{div}_x[\rho(R_0\chi)(\bar m_0)],\mathrm{div}_x[\rho\chi(\bar m_0)]).\end{aligned} \tag{3.19}$$

Once again, we emphasize the form (3.19), since the first and second-order term w.r.t. D_ρ are well identified, but the form (3.3) of the test-function φ is important to justify the existence of all the different terms with derivatives in x in (3.19), and the actual form of the latter is deduced, using (3.17)-(3.18), from (3.13):

$$\begin{aligned}\mathscr{L}\varphi(\rho) = \psi''(\langle\rho,\xi\rangle)\mathbb{E}\left[\langle\rho\chi(\bar m_0),\nabla_x\xi\rangle\langle\rho[R_1\chi+R_1R_0\chi](\bar m_0)],\nabla_x\xi\rangle\right]\\ +\psi'(\langle\rho,\xi\rangle)\mathbb{E}\Big[\langle\rho K(M)+\rho\chi(\bar m_0)\otimes(R_1R_0\chi)(\bar m_0),D_x^2\xi\rangle\\ +\langle\rho\chi(\bar m_0),\nabla_x(R_1R_0\chi)(\bar m_0)\nabla_x\xi\rangle\Big].\end{aligned}\tag{3.20}$$

By the resolvent formula $R_1R_0=R_0R_1=R_0-R_1$, again, we get

$$\begin{aligned}\mathscr{L}\varphi(\rho) = \psi''(\langle\rho,\xi\rangle)\mathbb{E}\left[\langle\rho\chi(\bar m_0),\nabla_x\xi\rangle\langle\rho(R_0\chi)(\bar m_0)],\nabla_x\xi\rangle\right]\\ +\psi'(\langle\rho,\xi\rangle)\mathbb{E}\Big[\langle\rho K(M)+\rho\chi(\bar m_0)\otimes(R_0R_1\chi)(\bar m_0),D_x^2\xi\rangle\\ +\langle\rho\chi(\bar m_0),\nabla_x(R_0R_1\chi)(\bar m_0)\nabla_x\xi\rangle\Big].\end{aligned}\tag{3.21}$$

Due to (3.13)-(3.21), the function $\Phi(f,n):=\mathscr{L}\varphi(\rho(f))-\mathscr{L}_\flat\varphi_1(f,n)$ satisfies the hypotheses of Proposition 2.9. We can define φ_2 thanks to the resolvent formula (2.36) and obtain a solution to (3.11). By Lemma 2.10, and examination of (3.13)-(3.21), we obtain a bound

$$|\varphi_2(f,n)|\leq\bar\Phi_2(\|\psi'\|_{C_b^2(\mathbb{R})},\|\xi\|_{C^2(\mathbb{T}^d)},\|f\|_{L^1(\mathbb{T}^d\times V)},\mathtt{R}),\tag{3.22}$$

where $\bar\Phi_2$ is an increasing function of its arguments and is bounded on bounded sets of $\mathbb{R}^4$.

3.1.3 Remainder and conclusion

Once φ_1 and φ_2 have been defined, the precise form of (3.1)-(3.2) is

$$\mathscr{L}^\varepsilon\varphi^\varepsilon(f,n)=\mathscr{L}\varphi(\rho)+\varepsilon\mathscr{L}_\flat\varphi_2(f,n).\tag{3.23}$$

Let us sum up the results of Section 3.1.1 and 3.1.2.

Proposition 3.1 (First corrector)**.** *Let $(\bar m_t)$ be an admissible pilot process in the sense of Definition 1.1. Let φ be defined by (3.3), let φ_1 and φ_2 be defined by the inversion, according to Proposition 2.9, of (3.4) and (3.11) respectively, with $\mathscr{L}\varphi(\rho)$ given by (3.21). Then φ_1 and φ_2 are in the domain of $\mathscr{L}_\sharp$ and $\mathscr{L}_\flat$ and satisfy the following bounds:*

$$|\varphi_1(f,n)|\leq\bar\Phi_1(\|\psi'\|_{C_b^1(\mathbb{R})},\|\xi\|_{C^1(\mathbb{T}^d)},\|f\|_{L^1(\mathbb{T}^d\times V)},\mathtt{R}),\tag{3.24}$$

$$|\varphi_2(f,n)|\leq\bar\Phi_2(\|\psi'\|_{C_b^2(\mathbb{R})},\|\xi\|_{C^2(\mathbb{T}^d)},\|f\|_{L^1(\mathbb{T}^d\times V)},\mathtt{R}),\tag{3.25}$$

where $\bar\Phi_1$, $\bar\Phi_2$ are some increasing functions of their arguments and are bounded on bounded sets of $\mathbb{R}^4$. For an analogous function $\bar\Phi_3$, we also have the bound

$$|\mathscr{L}_\flat\varphi_2(f,n)|\leq\bar\Phi_3(\|\psi'\|_{C_b^3(\mathbb{R})},\|\xi\|_{C^3(\mathbb{T}^d)},\|f\|_{L^1(\mathbb{T}^d\times V)},\mathtt{R}),\tag{3.26}$$

for all $f\in L^1(\mathbb{T}^d\times V)$ and $n\in F$ such that $\|n\|_F\leq\mathtt{R}$.

Proof of Proposition 3.1. We use Corollary 2.6. The estimate (3.24) is (3.10), (3.25) is (3.22). The estimate (3.26), which we will not prove in detail, follows from Lemma 2.10 and Lemma 2.11. □

We have the following corollary to Corollary 2.7 and Proposition 3.1.

Corollary 3.2. *Let φ be of the form (3.3), with $\xi \in C^3(\mathbb{T}^d)$ and ψ a Lipschitz function on $\mathbb{R}$ such that $\psi' \in C_b^\infty(\mathbb{R})$. Let φ_1, φ_2 be the correctors defined in Proposition 3.1. Let θ be the correction of φ at order 0, 1 or 2:*

$$\theta \in \{\varphi, \varphi + \varepsilon\varphi_1, \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2\}.$$

Then

$$M_\theta^\varepsilon(t) := \theta(f_t^\varepsilon, \bar{m}_t^\varepsilon) - \theta(f_{\mathrm{in}}, \bar{m}_0) - \int_0^t \mathscr{L}^\varepsilon \theta(f_s^\varepsilon, \bar{m}_s^\varepsilon) ds \tag{3.27}$$

is a martingale with quadratic variation given by

$$\langle M_\theta^\varepsilon, M_\theta^\varepsilon \rangle_t = \int_0^t \left[\mathscr{L}^\varepsilon |\theta|^2 - 2\theta \mathscr{L}^\varepsilon \theta\right] (f^\varepsilon(s), \bar{m}^\varepsilon(s)) ds, \tag{3.28}$$

for all $t \geq 0$.

Proof of Corollary 3.2. Let $(\mathcal{G}_t)$ be the filtration generated by $(\bar{m}_t)$ and let $\mathcal{G}_t^\varepsilon = \mathcal{G}_{\varepsilon^{-2}t}$. By (2.6), f_t^ε is $\mathcal{G}_t^\varepsilon$-measurable. The precise statement of Corollary, therefore, is that $(M_\theta^\varepsilon(t))$ is a $(\mathcal{G}_t^\varepsilon)$-martingale. The proof follows from Corollary 2.7 and [10, Appendix B]. □

3.2 Analysis of the limit generator

We want to do the analysis of $\mathscr{L}\varphi(\rho)$. In the first order terms of (3.19), we recognize the drift part of (1.11), with K^* and Ψ defined by (1.12) and (1.13). The second order term of (3.19) involves the operator S on $\left[L^2(\mathbb{T}^d)\right]^d$ with kernel $\mathtt{C}$ defined by

$$(Su)_i(x) = \sum_{j=1}^d \int_{\mathbb{T}^d} \mathtt{C}_{ij}(x,y) u_j(y) dy, \quad \mathtt{C}_{ij}(x,y) = \mathbb{E}\left[(R_0\chi_i)(\bar{m}_0)(x)\chi_j(\bar{m}_0)(y)\right]. \tag{3.29}$$

Indeed, if $\varphi(\rho) = \frac{1}{2}|\langle \rho, \xi\rangle|^2$, the second-order term in (3.19) is

$$\mathbb{E} D_\rho^2 \varphi(\rho) \cdot (\mathrm{div}_x[\rho(R_0\chi)(\bar{m}_0)], \mathrm{div}_x[\rho\chi(\bar{m}_0)]) = \langle S(\rho\nabla_x\xi), \rho\nabla_x\xi\rangle_{[L^2(\mathbb{T}^d)]^d}. \tag{3.30}$$

We will prove the following result.

Proposition 3.3. *Let $(\bar{m}_t)$ be an admissible pilot process in the sense of Definition 1.1. Then the operator S defined by (3.29) is bounded, symmetric and non-negative on $\left[L^2(\mathbb{T}^d)\right]^d$.*

The proof of Proposition 3.3 uses some identities similar to (3.17) and (3.18) and we begin with the proof of those two formulas.

Proof of (3.17) and (3.18). Regarding (3.17), we have

$$\begin{aligned}\mathbb{E}[J(\bar{g}_0)\Theta(\bar{m}_0)] &= \rho(g) \int_{-\infty}^0 e^t \mathbb{E}\left[\chi(\bar{m}_t)\Theta(\bar{m}_0)\right] dt \\ &= \rho(g) \int_{-\infty}^0 e^t \mathbb{E}\left[\chi(\bar{m}_0)\Theta(\bar{m}_{-t})\right] dt = \rho(g) \int_0^\infty e^{-t} \mathbb{E}\left[\chi(\bar{m}_0)\Theta(\bar{m}_t)\right] dt.\end{aligned}$$

We have used the fact that $(\bar{m}_t)$ is stationary. By conditioning on the σ-algebra generated by $\bar{m}_0$, we obtain

$$\int_0^\infty e^{-t}\mathbb{E}\left[\chi(\bar{m}_0)\Theta(\bar{m}_t)\right]dt = \mathbb{E}\left\{\chi(\bar{m}_0)\int_0^\infty e^{-t}\left[e^{tA}\Theta\right](\bar{m}_0)dt\right\}$$
$$= \mathbb{E}\left[\chi(\bar{m}_0)(R_1\Theta)(\bar{m}_0)\right].$$

To prove (3.18), we introduce $H(s) = \mathbb{E}\left[D^2_\rho\varphi(\rho)\cdot(\rho\chi(\bar{m}_0),\rho\chi(\bar{m}_s))\right]$. Note that H is even, $H(s) = H(-s)$, because $(\bar{m}_t)$ is a stationary process. We compute then

$$\int_{-\infty}^0\int_{-\infty}^0 e^{t+s}\mathbb{E}D^2_\rho\varphi(\rho)\cdot(\rho\chi(\bar{m}_t),\rho\chi(\bar{m}_s))dsdt = \int_{-\infty}^0\int_{-\infty}^0 e^{t+s}H(t-s)dsdt$$

By Fubini's Theorem and some elementary change of variables, this gives

$$\int_{-\infty}^0\int_{-\infty}^0 e^{t+s}\mathbb{E}D^2_\rho\varphi(\rho)\cdot(\rho\chi(\bar{m}_t),\rho\chi(\bar{m}_s))dsdt$$
$$= \int_{\sigma=0}^\infty e^{-\sigma}H(\sigma)d\sigma = \mathbb{E}D^2_\rho\varphi(\rho)\cdot(\rho\chi(\bar{m}_0),\rho(R_1\chi)(\bar{m}_0)).$$

□

Proof of Proposition 3.3. By (1.19) and (1.26), we have

$$|\mathtt{C}_{ij}(x,y)| \leq \mathtt{R}^2\|\gamma_{\mathrm{mix}}\|_{L^1(\mathbb{R}_+)}, \quad \|S\|_{[L^2(\mathbb{T}^d)]^d\to[L^2(\mathbb{T}^d)]^d} \leq \mathtt{R}^2\|\gamma_{\mathrm{mix}}\|_{L^1(\mathbb{R}_+)}. \tag{3.31}$$

That S is symmetric and non-negative follows from the following formula:

$$\mathtt{C}_{ij}(x,y) = \lim_{\alpha\to0^+}\alpha\int_{-\infty}^0\int_{-\infty}^0 e^{\alpha(t+s)}\mathbb{E}\left[\chi_i(\bar{m}_t)(x)\chi_j(\bar{m}_s)(y)\right]dtds. \tag{3.32}$$

To prove (3.32), we start from the fact that

$$\mathtt{C}_{ij}(x,y) = \lim_{\alpha\to0^+}\mathbb{E}\left[(R_\alpha\chi_i)(\bar{m}_0)(x)\chi_j(\bar{m}_0)(y)\right].$$

Then the proof is analogous to the proof of (3.18), with R_1 replaced by R_α. □

Being an operator with kernel $\mathtt{C}\in\left[L^\infty(\mathbb{T}^d)\right]^d\subset\left[L^2(\mathbb{T}^d)\right]^d$, the operator S is compact. By the spectral theorem, there exists an orthonormal basis (p_k) of $\left[L^2(\mathbb{T}^d)\right]^d$ and a sequence μ_k of non-negative real numbers such that $S = \sum_k \mu_k p_k\otimes p_k$. The square-root of S is then the operator $S^{1/2}$ defined by

$$S^{1/2} = \sum_k \mu_k^{1/2}p_k\otimes p_k. \tag{3.33}$$

In the spectral decomposition of S and in (3.33), we use the notation $p_k\otimes p_k$ to denote the projection operator $u\mapsto\langle u,p_k\rangle p_k$. Let $(\beta_k(t))_{k\in\mathbb{N}}$ be some independent one-dimensional Wiener processes. Consider the cylindrical Wiener process on $\left[L^2(\mathbb{T}^d)\right]^d$ defined by

$$W(t) = \sum_{k\in\mathbb{N}}\beta_k(t)p_k. \tag{3.34}$$

We have then the following Proposition.

Proposition 3.4. *Let $S^{1/2}$, $W(t)$ be defined by (3.33),(3.34) respectively. Then (1.11) defines a semi-group on $L^2(\mathbb{T}^d)$ with generator $\mathscr{L}$.*

The statement of Proposition 3.4 is somewhat imprecise. In Section 5, we explain what we mean, specifically, by "(1.11) defines a semi-group on $L^2(\mathbb{T}^d)$ with generator $\mathscr{L}$" and give the proof of Proposition 3.4. We end this section with the proof that $K^* \geq K(M)$ in the reversible case.

Proposition 3.5 (Enhanced diffusion)**.** *Let $(\bar{m}_t)$ be an admissible pilot process in the sense of Definition 1.1. Assume that $(\bar{m}_t)$ is reversible. Then the matrix K^* defined by (1.12) satisfies $K \geq K(M)$.*

Proof of Proposition 3.5. Let $\xi \in \mathbb{R}^d$ and let $\theta(n) = \chi(n) \cdot \xi$. We have to prove that

$$\langle R_0 R_1 \theta, \theta \rangle_{L^2(\lambda)} = \mathbb{E}\left[(R_0 R_1 \theta)(\bar{m}_0)\theta(\bar{m}_0)\right] \tag{3.35}$$

is non-negative. It is sufficient to prove

$$\langle R_\alpha R_\beta \theta, \theta \rangle_{L^2(\lambda)} \geq 0 \tag{3.36}$$

for $\alpha, \beta > 0$. By differentiation of the resolvent formula $R_{\alpha+h} R_\alpha = h^{-1}(R_\alpha - R_{\alpha+h})$, we obtain $\partial_\alpha R_\alpha = -R_\alpha^2$. When viewed as a function $\varphi(\alpha)$, the left-hand side of (3.36) has therefore the differential

$$\varphi'(\alpha) = -\langle R_\alpha^2 R_\beta \theta, \theta \rangle_{L^2(\lambda)} = -\langle R_\beta R_\alpha \theta, R_\alpha \theta \rangle_{L^2(\lambda)}. \tag{3.37}$$

To obtain the last identity in (3.37), we have used the fact that R_α is symmetric in $L^2(\lambda)$. Indeed, (e^{tA}) is symmetric in $L^2(\lambda)$ since $(\bar{m}(t))$ is reversible. The resolvent formula (1.24) shows that R_μ also. It follows from (3.37) that

$$\varphi'(\alpha) = -\beta \mathbb{E}\left|\int_{-\infty}^{0} e^{\beta s} R_\alpha \theta(\bar{m}_s) ds\right|^2 \leq 0. \tag{3.38}$$

The proof of (3.38) is similar to the proof of (3.32). We also have $\lim_{\alpha \to +\infty} \varphi(\alpha) = 0$, therefore $\varphi(\alpha) \geq 0$ for all $\alpha > 0$, which is the desired result. □

4 Tightness

4.1 Bound in L^1

Assume $f_{\text{in}}^\varepsilon \geq 0$ a.e. By (1.19), (1.20), the positivity hypothesis (2.8) is satisfied a.s.:

$$M(v) + v \cdot \nabla_x \bar{m}_t(x) \geq 0$$

a.e. on $\mathbb{T}^d \times V \times \mathbb{R} \times \Omega$. The solutions to (1.1) also satisfy the conservation

$$\iint_{\mathbb{T}^d \times V} f^\varepsilon(t, x, v) dx d\nu(v) = cte.$$

It follows, assuming (1.9), that we have the first uniform estimate

$$\|f^\varepsilon(t)\|_{L^1(\mathbb{T}^d \times V)} \leq C_{\text{in}} \text{ a.s.}, \tag{4.1}$$

for all $t \geq 0$. Note that, V being bounded by hypothesis, (4.1) gives a bound on all the moments in v of $f^\varepsilon(t)$. Actually, the bound

$$\||v|^m f\|_{L^1(\mathbb{T}^d \times V)} \leq \|f\|_{L^1(\mathbb{T}^d \times V)}$$

has already been used in the derivation of (3.24), (3.25), (3.26).

4.2 Relative entropy estimate

Let $\bar{M}_t$ be defined by (2.34). Let $\bar{M}^\varepsilon_t = \bar{M}_{\varepsilon^{-2}t}$. We consider the relative entropy

$$\mathcal{H}^\varepsilon(t) := \mathcal{H}(f^\varepsilon(t)|\bar{M}^\varepsilon_t) := \iint_{\mathbb{T}^d\times V} H\left(\frac{f^\varepsilon(x,t,v)}{\bar{M}^\varepsilon_t(x,v)}\right)\bar{M}^\varepsilon_t(x,v)dxd\nu(v), \tag{4.2}$$

where H is the square function

$$H(u) = \frac{u^2}{2}. \tag{4.3}$$

We also introduce the dissipation term

$$\mathcal{D}^\varepsilon(t) = \iint_{\mathbb{T}^d\times V} \frac{|L^\varepsilon_t f^\varepsilon_t(x,v)|^2}{\bar{M}^\varepsilon_t(x,v)}dxd\nu(v), \tag{4.4}$$

where L^ε_t is the operator

$$L^\varepsilon_t f = \rho(f)\bar{M}^\varepsilon_t - f. \tag{4.5}$$

We will prove the following estimate.

Proposition 4.1 (Relative entropy estimate)**.** *Let $f^\varepsilon_{\text{in}} \in L^2(\mathbb{T}^d\times V)$. Let M satisfy (1.5). Let $(\bar{m}_t)$ be an admissible pilot process in the sense of Definition 1.1. Then the mild solution f^ε_t to (1.1) with initial datum $f^\varepsilon_{\text{in}}$ satisfies the relative entropy estimate*

$$\mathcal{H}^\varepsilon(t) + \frac{1}{2\varepsilon^2}\int_0^t \mathcal{D}^\varepsilon(s)ds \leq e^t\mathcal{H}^\varepsilon(0), \tag{4.6}$$

almost-surely, for every $t\geq 0$.

Proof of Proposition 4.1. Introduce the operator

$$\check{L}^\varepsilon_t f = \rho(f)\check{M}^\varepsilon_t - f, \quad \check{M}^\varepsilon_t = M + v\cdot\nabla_x\bar{m}^\varepsilon_t. \tag{4.7}$$

Using first Equation (1.1) for f^ε_t, which reads,

$$\partial_t f^\varepsilon_t + \frac{v}{\varepsilon}\cdot\nabla_x f^\varepsilon_t = \frac{1}{\varepsilon^2}\check{L}^\varepsilon_t f^\varepsilon_t,$$

using, secondly, the equation for the reference solution $\bar{M}^\varepsilon_t$

$$\partial_t\bar{M}^\varepsilon_t = \frac{1}{\varepsilon^2}\check{L}^\varepsilon_t\bar{M}^\varepsilon_t, \tag{4.8}$$

and Proposition 2.3 to justify the following computations, we obtain the decomposition

$$\frac{d}{dt}\mathcal{H}^\varepsilon(t) = -\frac{1}{\varepsilon^2}A^\varepsilon_t + \frac{1}{\varepsilon}B^\varepsilon_t, \tag{4.9}$$

where

$$A^\varepsilon_t = -\iint_{\mathbb{T}^d\times V}\left[\frac{f^\varepsilon_t}{\bar{M}^\varepsilon_t}\check{L}^\varepsilon_t f^\varepsilon_t - \frac{1}{2}\frac{|f^\varepsilon_t|^2}{|\bar{M}^\varepsilon_t|^2}\check{L}^\varepsilon_t\bar{M}^\varepsilon_t\right]dxd\nu(v),$$

and, after integration by parts,

$$B^\varepsilon_t = -\frac{1}{2}\iint_{\mathbb{T}^d\times V}\frac{|f^\varepsilon_t|^2}{|\bar{M}^\varepsilon_t|^2}v\cdot\nabla_x\bar{M}^\varepsilon_t dxd\nu(v).$$

Note that

$$\check{L}_t^\varepsilon f = L_t^\varepsilon f + \rho(f)(\check{M}_t^\varepsilon - \bar{M}_t^\varepsilon), \quad \check{L}_t^\varepsilon \bar{M}_t^\varepsilon = \check{M}_t^\varepsilon - \bar{M}_t^\varepsilon,$$

and thus

$$A_t^\varepsilon = -\iint_{\mathbb{T}^d\times V} \frac{f_t^\varepsilon}{\bar{M}_t^\varepsilon} L_t^\varepsilon f_t^\varepsilon dxd\nu(v) + \iint_{\mathbb{T}^d\times V} \left[\frac{1}{2}\frac{|f_t^\varepsilon|^2}{|\bar{M}_t^\varepsilon|^2} - \rho_t^\varepsilon \frac{f_t^\varepsilon}{\bar{M}_t^\varepsilon}\right](\check{M}_t^\varepsilon - \bar{M}_t^\varepsilon)dxd\nu(v). \quad (4.10)$$

We use the identities

$$\iint_{\mathbb{T}^d\times V} \rho_t^\varepsilon L_t^\varepsilon f_t^\varepsilon dxd\nu(v) = 0, \quad \iint_{\mathbb{T}^d\times V} \rho_t^\varepsilon (\check{M}_t^\varepsilon - \bar{M}_t^\varepsilon)dxd\nu(v) = 0$$

to rewrite the right-hand side of (4.10) as follows:

$$\begin{aligned} A_t^\varepsilon &= \iint_{\mathbb{T}^d\times V} \frac{|L_t^\varepsilon f_t^\varepsilon|^2}{\bar{M}_t^\varepsilon} dxd\nu(v) + \iint_{\mathbb{T}^d\times V} \left[\frac{1}{2}\frac{|f_t^\varepsilon|^2}{|\bar{M}_t^\varepsilon|^2} - \rho_t^\varepsilon \frac{f_t^\varepsilon}{\bar{M}_t^\varepsilon} + \frac{1}{2}|\rho_t^\varepsilon|^2\right](\check{M}_t^\varepsilon - \bar{M}_t^\varepsilon)dxd\nu(v) \\ &= \iint_{\mathbb{T}^d\times V} \frac{|L_t^\varepsilon f_t^\varepsilon|^2}{\bar{M}_t^\varepsilon} dxd\nu(v) + \iint_{\mathbb{T}^d\times V} \frac{|L_t^\varepsilon f_t^\varepsilon|^2}{\bar{M}_t^\varepsilon}\frac{\check{M}_t^\varepsilon - \bar{M}_t^\varepsilon}{2\bar{M}_t^\varepsilon} dxd\nu(v). \end{aligned}$$

Eventually, this gives

$$A_t^\varepsilon = \iint_{\mathbb{T}^d\times V} \frac{|L_t^\varepsilon f_t^\varepsilon|^2}{\bar{M}_t^\varepsilon}\frac{\check{M}_t^\varepsilon + \bar{M}_t^\varepsilon}{\bar{M}_t^\varepsilon} dxd\nu(v) \geq \mathcal{D}^\varepsilon(t). \quad (4.11)$$

Indeed, $\check{M}_t^\varepsilon \geq 0$ a.e. due to (1.5), (1.19), (1.20). In the second term B_t^ε, we can decompose $f_t^\varepsilon = \rho_t^\varepsilon \bar{M}_t^\varepsilon - L_t^\varepsilon f_t^\varepsilon$. Since $|v\cdot\nabla_x \bar{M}_t^\varepsilon| \leq \mathtt{R}$ by (1.19) and $\alpha - \mathtt{R} \leq \bar{M}_t^\varepsilon$ by (1.5)-(1.19), this gives

$$\frac{1}{\varepsilon}|B_t^\varepsilon| \leq \frac{1}{\varepsilon^2}\frac{\mathtt{R}}{\alpha-\mathtt{R}}\mathcal{D}^\varepsilon(t) + \int_{\mathbb{T}^d} |\rho_t^\varepsilon|^2 dx.$$

By (1.20), we have $\frac{\mathtt{R}}{\alpha-\mathtt{R}} \leq \frac{1}{2}$. By (4.9) and (4.11), we obtain

$$\frac{d}{dt}\mathcal{H}^\varepsilon(t) \leq -\frac{1}{2\varepsilon^2}\mathcal{D}^\varepsilon(t) + \int_{\mathbb{T}^d} |\rho_t^\varepsilon|^2 dx. \quad (4.12)$$

The Cauchy-Schwarz inequality applied to the product of $[\bar{M}_t^\varepsilon]^{-1/2} f_t^\varepsilon$ with $[\bar{M}_t^\varepsilon]^{-1/2}$ gives also the inequality

$$\int_{\mathbb{T}^d} |\rho_t^\varepsilon|^2 dx \leq \iint_{\mathbb{T}^d\times V} \frac{|f_t^\varepsilon|^2}{\bar{M}_t^\varepsilon} dxd\nu(v) = \mathcal{H}^\varepsilon(t). \quad (4.13)$$

Combined with (4.12), and the Gronwall Lemma, this gives (4.6). □

Assume that the uniform L^2-bound in (1.9) is satisfied. Two important corollaries from (4.6) are then, first, using (1.5), the estimate (1.10) and, second, using (4.13), the bound

$$\|\rho_t^\varepsilon\|_{L^2(\mathbb{T}^d\times V)}^2 \leq \alpha^{-2} C_{\mathrm{in}}^2, \quad (4.14)$$

almost-surely, which gives a uniform L^2-estimate on ρ_t^ε. We use these bounds in the next section 4.3 to obtain the tightness of (ρ_t^ε) in the space $C([0,T]; H^{-\sigma}(\mathbb{T}^d))$.

4.3 Time regularity

In this section, we will prove that, for any $\sigma > 0$, the sequence (of the laws of) (ρ_t^ε) is tight in $C([0,T]; H^{-\sigma}(\mathbb{T}^d))$. To obtain this result, we will establish some estimates in the Hölder space $C^\delta([0,T]; H^{-\sigma}(\mathbb{T}^d))$ on a perturbation to ρ_t^ε. First, let us set some notations: for $\sigma \geq 0$, $H^{-\sigma}(\mathbb{T}^d)$ is the dual space of $H^\sigma(\mathbb{T}^d)$. Recall also that, by the Sobolev embedding, for all $k \in \mathbb{N}$, for all $\sigma > k + \frac{d}{2}$, there is a constant $C(k,\sigma) \geq 0$ such that

$$\|\xi\|_{C^k(\mathbb{T}^d)} \leq C(k,\sigma)\|\xi\|_{H^\sigma(\mathbb{T}^d)}, \tag{4.15}$$

for all $\xi \in H^\sigma(\mathbb{T}^d)$ (where the function ξ in the left-hand side of (4.15) is a specific member of the equivalent class, for equality a.e., of $\xi \in H^\sigma(\mathbb{T}^d)$).

Proposition 4.2 (Tightness - 2)**.** *Let $f_{\text{in}}^\varepsilon \in L^2(\mathbb{T}^d \times V)$. Let $(\bar{m}_t)$ be an admissible pilot process in the sense of Definition 1.1. Let f^ε be the mild solution to (1.1) with initial datum $f_{\text{in}}^\varepsilon$. Assume that f^ε and M satisfy (1.9) and (1.5) respectively. Then $(\rho_t^\varepsilon)_{t\in[0,T]}$ is tight in the space $C([0,T]; H^{-\sigma}(\mathbb{T}^d))$.*

Proof of Proposition 4.2. Thanks to the uniform estimate (4.14), it is sufficient to prove the result for σ large. We begin by assuming $\sigma > 1 + \frac{d}{2}$ and let $\xi \in H^\sigma(\mathbb{T}^d)$. We are interested in the quantity

$$\varphi(\rho_t^\varepsilon) = \langle \rho_t^\varepsilon, \xi \rangle.$$

In view of the expression (3.8) of the first corrector in the perturbed test-function method, we introduce the perturbation

$$\zeta_t^\varepsilon = \rho_t^\varepsilon - \varepsilon \operatorname{div}_x(J(f_t^\varepsilon) + \rho_t^\varepsilon (R_0\chi)(\bar{m}_t^\varepsilon)). \tag{4.16}$$

We show first that ρ^ε is close to ζ^ε in $C([0,T]; H^{-\sigma}(\mathbb{T}^d))$ and then prove that (ζ^ε) is tight in $C([0,T]; H^{-\sigma}(\mathbb{T}^d))$.

Step 1. ρ^ε is close to ζ^ε. We show that the difference $\rho_t^\varepsilon - \zeta_t^\varepsilon$ is of order ε in the space $C([0,T]; H^{-\sigma}(\mathbb{T}^d))$. More precisely, due to (1.26), (1.9), (4.1) and (4.16), we have: a.s., for all $t \in [0,T]$,

$$|\langle \rho_t^\varepsilon - \zeta_t^\varepsilon, \xi \rangle| \leq (1 + \mathtt{R}\|\gamma_{\text{mix}}\|_{L^1(\mathbb{R}_+)}) C_{\text{in}} \varepsilon \|\nabla_x \xi\|_{C(\mathbb{T}^d)}. \tag{4.17}$$

Note that ρ_t^ε and ζ_t^ε are both continuous in time with values in $H^{-\sigma}(\mathbb{T}^d)$ due to the continuity of f^ε with values in $L^1(\mathbb{T}^d \times V)$. By (4.15) with $k = 1$, we deduce from (4.17) that

$$\sup_{t\in[0,T]} \|\rho_t^\varepsilon - \zeta_t^\varepsilon\|_{H^{-\sigma}(\mathbb{T}^d)} \leq (1 + \mathtt{R}\|\gamma_{\text{mix}}\|_{L^1(\mathbb{R}_+)}) C_{\text{in}} \varepsilon, \quad \mathbb{P} - \text{a.s.} \tag{4.18}$$

Step 2. (ζ^ε) is tight in $C([0,T]; H^{-\sigma}(\mathbb{T}^d))$. For $M > 0$, $\delta \in (0, 3/4)$, $\sigma > \sigma_0$ with $\sigma_0 > 1 + \frac{d}{2}$, define the set

$$K_M = \left\{ \zeta \in C([0,T]; H^{-\sigma}(\mathbb{T}^d)); \|\zeta\|_{\dot{C}^\delta([0,T]; H^{-\sigma}(\mathbb{T}^d))} + \|\zeta\|_{C([0,T]; H^{-\sigma_0}(\mathbb{T}^d))} \leq M \right\},$$

where

$$\|\zeta\|_{\dot{C}^\delta([0,T]; H^{-\sigma}(\mathbb{T}^d))} = \sup_{s\neq t\in[0,T]} \frac{\|\zeta(t) - \zeta(s)\|_{H^{-\sigma}(\mathbb{T}^d)}}{|t-s|^\delta}. \tag{4.19}$$

By the Arzéla-Ascoli Theorem, the set K_M is compact in $C([0,T]; H^{-\sigma}(\mathbb{T}^d))$. By the same argument as in Step 1, we have

$$\|\zeta^\varepsilon\|_{C([0,T]; H^{-\sigma_0}(\mathbb{T}^d))} \leq (1 + \mathtt{R}\|\gamma_{\text{mix}}\|_{L^1(\mathbb{R}_+)}) C_{\text{in}}, \quad \mathbb{P} - \text{a.s.} \tag{4.20}$$

We will show furthermore that

$$\mathbb{E}\|\zeta^\varepsilon\|_{\dot{C}^\delta([0,T];H^{-\sigma}(\mathbb{T}^d))} \leq C_1, \tag{4.21}$$

where by C_1, and by C_2, C_3... in what follows, we denote some constant depending on the dimension d, on the constant α in (1.5), on the constant $C_{\rm in}$ in (1.9), on $\mathtt{R}$, on the constant $C^0_{\mathtt{R}}$ in (1.28), but not on ε. Not that (4.20), (4.21) and the Markov inequality yield, for M larger than twice the right-hand side of (4.20),

$$\mathbb{P}(\zeta^\varepsilon \notin K_M) \leq \mathbb{P}\left(\|\zeta\|_{\dot{C}^\delta([0,T];H^{-\sigma}(\mathbb{T}^d))} > \frac{M}{2}\right) \leq \frac{2C_1}{M},$$

which shows that (ζ^ε) is tight in $C([0,T];H^{-\sigma}(\mathbb{T}^d))$. To establish (4.21), we will use the fact that

$$\|\Lambda\|^2_{H^{-\sigma}(\mathbb{T}^d)} = \sum_{k\in\mathbb{Z}^d}(1+4\pi^2|k|^2)^{-\sigma}|\langle\Lambda,w_k\rangle|^2, \quad w_k := \exp(2\pi i k\cdot x), \tag{4.22}$$

where $\langle\Lambda,w_k\rangle$ denotes the duality product $\langle\Lambda,w_k\rangle_{H^{-\sigma}(\mathbb{T}^d),H^\sigma(\mathbb{T}^d)}$. We will also use the Kolmogorov's continuity criterion, through the Garsia - Rodemich - Rumsey inequality, [2, Theorem 7.34]: for $q>1$, $a\in(q^{-1},1)$,

$$\|\zeta(t)-\zeta(s)\|^q_{H^{-\sigma}(\mathbb{T}^d)} \leq C_2|s-t|^{aq-1}\int_0^T\int_0^T \frac{\|\zeta(r)-\zeta(\tau)\|^q_{H^{-\sigma}(\mathbb{T}^d)}}{|r-\tau|^{aq+1}}drd\tau. \tag{4.23}$$

We will apply (4.23) with $q=4$, $a=\frac{1}{4}+\delta$ (remember that $\delta\in(0,\frac{3}{4})$). The estimate (4.21) will be a consequence of the bound

$$\mathbb{E}|\langle\zeta^\varepsilon_t-\zeta^\varepsilon_s,\xi\rangle|^4 \leq C_3|t-s|^2\|\xi\|^4_{H^{\sigma_0}(\mathbb{T}^d)}, \tag{4.24}$$

for all $\xi\in H^{\sigma_0}(\mathbb{T}^d)$, where

$$\sigma > \sigma_0+\frac{d}{2}, \quad \sigma_0 > 2+\frac{d}{2}. \tag{4.25}$$

Indeed, (4.24), (4.22) and the Cauchy-Schwarz inequality ($\theta>0$ will be fixed later)

$$\left(\sum_{k\in\mathbb{Z}^d}(1+4\pi^2|k|^2)^{-\sigma}|\langle\Lambda,w_k\rangle|^2\right)^2 \leq \sum_{k\in\mathbb{Z}^d}(1+4\pi^2|k|^2)^{-2(\sigma-\theta)}\cdot\sum_{k\in\mathbb{Z}^d}(1+4\pi^2|k|^2)^{-2\theta}|\langle\Lambda,w_k\rangle|^4$$

give

$$\mathbb{E}\|\zeta(r)-\zeta(\tau)\|^4_{H^{-\sigma}(\mathbb{T}^d)} \leq C_3|r-\tau|^4, \tag{4.26}$$

where the constant

$$C_4 = C_3\sum_{k\in\mathbb{Z}^d}(1+4\pi^2|k|^2)^{-2(\sigma-\theta)}\cdot\sum_{k\in\mathbb{Z}^d}(1+4\pi^2|k|^2)^{-2(\theta-\sigma_0)}$$

is finite as soon as

$$\theta > \sigma_0+\frac{d}{4} \quad\text{and}\quad \sigma > \theta+\frac{d}{4}. \tag{4.27}$$

Under the first inequality in (4.25), both constraints (4.27) are realized for a given θ between $\sigma_0+\frac{d}{4}$ and σ. Then (4.23) yields (4.21) with

$$C_1 = C_2C_4\int_0^T\int_0^T|r-\tau|^{4\delta-1}drd\tau,$$

which is finite since $\delta > 0$. To obtain the estimate (4.24) on the time increments of $\langle \zeta^\varepsilon, \xi \rangle$, we introduce the process

$$M_t^\varepsilon = \varphi^\varepsilon(f_t^\varepsilon, \bar{m}_t^\varepsilon) - \varphi^\varepsilon(f_{\mathrm{in}}^\varepsilon, \bar{m}_0^\varepsilon) - \int_0^t \mathscr{L}^\varepsilon \varphi^\varepsilon(f_s^\varepsilon, \bar{m}_s^\varepsilon) ds. \tag{4.28}$$

In (4.28), φ^ε is the first-order correction $\varphi^\varepsilon := \varphi + \varepsilon\varphi_1$ of the function $\varphi(\rho) = \langle \rho, \xi \rangle$. By (3.8), we have precisely $\varphi^\varepsilon(f_t^\varepsilon, \bar{m}_t^\varepsilon) = \langle \zeta_t^\varepsilon, \xi \rangle$ and thus

$$\langle \zeta_t^\varepsilon - \zeta_s^\varepsilon, \xi \rangle = \int_s^t \mathscr{L}^\varepsilon \varphi^\varepsilon(f_s^\varepsilon, \bar{m}_s^\varepsilon) ds + M_t^\varepsilon - M_s^\varepsilon. \tag{4.29}$$

Observe that $\mathscr{L}^\varepsilon \varphi^\varepsilon = \mathscr{L}_\flat \varphi_1$. A bound, which is more simple to establish in this linear context, but analogous to (3.22), gives

$$|\mathscr{L}_\flat \varphi_1(f, n)| \leq C_5 \|f\|_{L^1(\mathbb{T}^d \times V)} \|\xi\|_{C^2(\mathbb{T}^d)}.$$

By (4.1), (4.15) and the second inequality in (4.25), we deduce that.

$$\mathbb{E}\left| \int_s^t \mathscr{L}^\varepsilon \varphi^\varepsilon(f_s^\varepsilon, \bar{m}_s^\varepsilon) ds \right|^4 \leq C_5 |t - s|^4 \|\xi\|_{H^{\sigma_0}(\mathbb{T}^d)}^4. \tag{4.30}$$

In analogy with Corollary 2.7, the process (M_t^ε) is a martingale with quadratic variation

$$\langle M^\varepsilon, M^\varepsilon \rangle_t = \int_0^t \left[\mathscr{L}^\varepsilon |\varphi^\varepsilon|^2 - 2\varphi^\varepsilon \mathscr{L}^\varepsilon \varphi^\varepsilon \right] (f_s^\varepsilon, \bar{m}_s^\varepsilon) ds.$$

Let us compute exactly this quadratic variation: since $D_f |\varphi^\varepsilon|^2 - 2\varphi^\varepsilon D_f \varphi^\varepsilon = 0$, only the part $\varepsilon^{-2} A$ of the generator $\mathscr{L}^\varepsilon$ is contributing to the quadratic variation. Since, in addition, $A|\varphi|^2 = 0$, $A\varphi = 0$, we obtain

$$\langle M^\varepsilon, M^\varepsilon \rangle_t = \int_0^t \left[A|\varphi_1|^2 - 2\varphi_1 A\varphi_1 \right] (f^\varepsilon(s), \bar{m}^\varepsilon(s)) ds. \tag{4.31}$$

Since $\varphi_1(f, n) = c + \Lambda(R_0 \chi(n))$, where $\Lambda(\chi) = \langle \rho(f)\chi, \nabla\xi \rangle$, we have

$$\left[A|\varphi_1|^2 - 2\varphi_1 A\varphi_1 \right] (f, n) = A|\Lambda(R_0\chi(n))|^2 - 2\Lambda(R_0\chi(n))) A\Lambda(R_0\chi(n)).$$

Since $\|\Lambda\| \leq \|f\|_{L^1(\mathbb{T}^d \times V)} \|\xi\|_{C^1(\mathbb{T}^d)}$, the estimates (1.28), (4.1) and (4.15) give

$$\left| \left[A|\varphi_1|^2 - 2\varphi_1 A\varphi_1 \right] (f^\varepsilon(s), \bar{m}^\varepsilon(s)) \right| \leq C_6 \|\xi\|_{H^{\sigma_0}(\mathbb{T}^d)}^2, \tag{4.32}$$

almost-surely. We deduce from (4.31) that

$$\mathbb{E}|\langle M^\varepsilon, M^\varepsilon \rangle_t - \langle M^\varepsilon, M^\varepsilon \rangle_s|^2 \leq C_7 |t - s|^2 \|\xi\|_{H^{\sigma_0}(\mathbb{T}^d)}^4 \tag{4.33}$$

We use the Burkholder - Davis - Gundy inequality, [4], to get the following bound

$$\mathbb{E}|M_t^\varepsilon - M_s^\varepsilon|^4 \leq C_8 |t - s|^2 \|\xi\|_{H^{\sigma_0}(\mathbb{T}^d)}^4. \tag{4.34}$$

The two estimates (4.30), (4.34) and the decomposition (4.29) yield (4.24). □

4.4 Convergence to the solution of a Martingale problem

Assume that the hypotheses of Proposition 4.2 are satisfied. Let (ε_n) be a sequence which decreases to 0 and let $\sigma > 0$. Set $\varepsilon_{\mathbb{N}} = \{\varepsilon_n; n \in \mathbb{N}\}$. By the Skorohod theorem [3, p. 70], there is a subset of $\varepsilon_{\mathbb{N}}$, which we still denote by $\varepsilon_{\mathbb{N}}$, a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, some random variables $\{\tilde{\rho}^\varepsilon; \varepsilon \in \varepsilon_{\mathbb{N}}\}$, $\tilde{\rho}$ on $C([0,T]; H^{-\sigma}(\mathbb{T}^d))$, such that

1. for all $\varepsilon \in \varepsilon_{\mathbb{N}}$, the laws of ρ^ε and $\tilde{\rho}^\varepsilon$ as $C([0,T]; H^{-\sigma}(\mathbb{T}^d))$-random variables coincide,
2. $\tilde{\mathbb{P}}$-a.s., $(\tilde{\rho}^\varepsilon)$ is converging to $\tilde{\rho}$ in $C([0,T]; H^{-\sigma}(\mathbb{T}^d))$ along $\varepsilon_{\mathbb{N}}$.

Let $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}$ be the natural filtration of $(\tilde{\rho}(t))_{t\in[0,T]}$. Our aim is to show that the process $(\tilde{\rho}(t))_{t\in[0,T]}$ is solution of the martingale problem associated to the limit generator $\mathscr{L}$.

Proposition 4.3 (Martingale)**.** *Let $\sigma \in (0,1)$, $\xi \in H^{\sigma+2}(\mathbb{T}^d)$, and let φ be defined by $\varphi(\rho) = \psi\left(\langle \rho, \xi\rangle_{H^{-\sigma},H^\sigma}\right)$, where ψ is a Lipschitz function on $\mathbb{R}$ such that $\psi' \in C_b^\infty(\mathbb{R})$. Then the process*

$$\tilde{M}_\varphi(t) := \varphi(\tilde{\rho}(t)) - \varphi(\tilde{\rho}(0)) - \int_0^t \mathscr{L}\varphi(\tilde{\rho}(s))ds \tag{4.35}$$

is a continuous martingale with respect to $(\tilde{\mathcal{F}}_t)_{t\in[0,T]}$.

Proof of Proposition 4.3. Let $0 \le s \le t \le T$. Let $0 \le t_1 < \cdots < t_n \le s$ and let Θ be a continuous and bounded function on $[H^{-\sigma}(\mathbb{T}^d)]^n$. Note that $\tilde{\mathcal{F}}_s$ is generated by the random variables $\Theta(\tilde{\rho}(t_1), \dots, \tilde{\rho}(t_n))$, for $n \in \mathbb{N}^*$, $(t_i)_{1,n}$ and Θ as above. Our aim is therefore to prove that

$$\mathbb{E}\left[(\tilde{M}_\varphi(t) - \tilde{M}_\varphi(s))\Theta(\tilde{\rho}(t_1), \dots, \tilde{\rho}(t_n))\right] = 0. \tag{4.36}$$

Let $\varphi^\varepsilon = \varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2$ be the second order correction of φ, with φ_1 and φ_2 given by Proposition 3.1. We use Corollary 3.2:

$$\mathbb{E}\left[(M^\varepsilon_\varphi(t) - M^\varepsilon_\varphi(s))\Theta(\rho^\varepsilon(t_1), \dots, \rho^\varepsilon(t_n))\right] = 0, \tag{4.37}$$

where

$$M^\varepsilon_\varphi(t) := \varphi^\varepsilon(f^\varepsilon(t), \bar{m}^\varepsilon_t) - \varphi^\varepsilon(f_{\mathrm{in}}, \bar{m}^\varepsilon_0) - \int_0^t \mathscr{L}^\varepsilon\varphi^\varepsilon(f^\varepsilon(s), \bar{m}^\varepsilon_s)ds. \tag{4.38}$$

Recall that $\mathscr{L}^\varepsilon\varphi^\varepsilon = \mathscr{L}\varphi + \varepsilon\mathscr{L}_\flat\varphi_2$. By (4.37), using the estimate (3.26) on the remainder term $\mathscr{L}_\flat\varphi_2$ and the L^1-estimate (4.1), we have

$$\mathbb{E}\left[(X^\varepsilon_\varphi(t) - X^\varepsilon_\varphi(s))\Theta(\rho^\varepsilon(t_1), \dots, \rho^\varepsilon(t_n))\right] = \mathcal{O}(\varepsilon),$$

where the process $(X^\varepsilon_\varphi(t))$ is

$$X^\varepsilon_\varphi(t) = \varphi(\rho^\varepsilon(t)) - \varphi(\rho_{\mathrm{in}}) - \int_0^t \mathscr{L}\varphi(\rho^\varepsilon(s))ds.$$

By identities of the laws, it follows that

$$\tilde{\mathbb{E}}\left[\left(\varphi(\tilde{\rho}^\varepsilon(t)) - \varphi(\tilde{\rho}^\varepsilon(s)) - \int_s^t \mathscr{L}\varphi(\tilde{\rho}^\varepsilon(s))ds\right)\Theta(\tilde{\rho}^\varepsilon(t_1), \dots, \tilde{\rho}^\varepsilon(t_n))\right] = \mathcal{O}(\varepsilon). \tag{4.39}$$

We must examine the convergence of each terms in (4.39). By a.s convergence of $(\tilde{\rho}^\varepsilon)$ in $C([0,T];H^{-\sigma}(\mathbb{T}^d))$ along $\varepsilon_\mathbb{N}$, we have

$$\left[\varphi(\tilde{\rho}^\varepsilon(t)) - \int_0^t \mathscr{L}\varphi(\tilde{\rho}^\varepsilon(s))ds\right]\Theta(\tilde{\rho}^\varepsilon(t_1),\dots,\tilde{\rho}^\varepsilon(t_n)) \\ \to \left[\varphi(\tilde{\rho}(t)) - \int_0^t \mathscr{L}\varphi(\tilde{\rho}(s))ds\right]\Theta(\tilde{\rho}(t_1),\dots,\tilde{\rho}(t_n))$$

almost-surely when $\varepsilon \to 0$ along $\varepsilon_\mathbb{N}$. Indeed, $\mathscr{L}\varphi$ is continuous on $H^{-\sigma}(\mathbb{T}^d)$ by (3.21) and the fact that $\xi \in H^{\sigma+2}(\mathbb{T}^d)$, $\bar{m_0} \in C^2(\mathbb{T}^d)$. Since Θ is bounded and $\varphi(\tilde{\rho}^\varepsilon(t))$ and $\mathscr{L}\varphi(\tilde{\rho}^\varepsilon(t))$ are a.s. bounded by a constant, we can apply the dominated convergence theorem. This gives (4.36). □

5 The limit equation

The analysis of the limit generator $\mathscr{L}$ was initiated in Section 3.2. We complete this study here, in particular we establish the relation between $\mathscr{L}$ and the stochastic PDE (1.11), and use the good properties of the limit equation (1.11) to conclude the proof of convergence of f^ε.

5.1 Resolution of the limit equation

Let $S^{1/2}$ and $W(t)$ be defined by (3.33) and (3.34) respectively. By testing the limit equation (1.11) against a smooth function ξ, we obtain the one-dimensional SDE

$$d\langle\rho_t,\xi\rangle_{H^{-\sigma},H^\sigma} = b(\rho_t,\xi)dt + \sum_k \sigma_k(\rho_t,\xi)d\beta_k(t),$$

with coefficients

$$b(\rho,\xi) = \langle\rho, \mathrm{div}_x(K^*\nabla_x\xi) - \Psi\nabla_x\xi\rangle_{H^{-\sigma},H^\sigma}, \quad \sigma_k(\rho,\xi) = -\sqrt{2}\mu_k^{1/2}\langle\rho\nabla_x\xi, p_k\rangle_{L^2(\mathbb{T}^d)}. \tag{5.1}$$

Definition 5.1. Let $\rho_{\mathrm{in}} \in L^2(\mathbb{T}^d)$. Let $W(t)$ be given by (3.34), let $(\mathcal{F}_t^W)$ be the filtration generated by W. A $(\mathcal{F}_t^W)$-adapted process $\rho \in C([0,T];L^2(\mathbb{T}^d))$ is said to be a weak solution to (1.11) in $L^2(\mathbb{T}^d)$, with initial datum ρ_{in}, if $\rho \in L^2(0,T;H^1(\mathbb{T}^d))$ almost-surely and

$$\langle\rho_t,\xi\rangle_{H^{-\sigma},H^\sigma} = \langle\rho_{\mathrm{in}},\xi\rangle_{H^{-\sigma},H^\sigma} + \int_0^t b(\rho_s,\xi)ds + \sum_{k\in\mathbb{N}}\int_0^t \sigma_k(\rho_s,\xi)d\beta_k(s), \tag{5.2}$$

for all $\xi \in C^1(\mathbb{T}^d)$, for all $t \in [0,T]$, where b and σ_k are defined in (5.1).

Theorem 5.1. *Let $S^{1/2}$ be the Hilbert-Schmidt operator defined by (3.33). Let $W(t)$ be given by (3.34). Then there exists a unique weak solution to (1.11) in $L^2(\mathbb{T}^d)$ with initial datum ρ_{in}.*

Proof of Theorem 5.1. We proceed as in [10], *proof of Theorem 6.9.* The essential step is to prove an energy estimate for weak solutions to (1.11). It gives uniqueness, and existence via approximation (*e.g.* Galerkin's approximation). □

5.2 Conclusion

To conclude the proof of Theorem 1.1, we show the two following facts:

1. the resolution of (1.11) according to Theorem 5.1 defines a Markov semi-group on $L^2(\mathbb{T}^d)$ with generator $\mathscr{L}$,
2. there is uniqueness of the martingale problem associated to $\mathscr{L}$.

Then we use the martingale property of Proposition 4.3 to show that $\tilde{\rho}_t$ coincides, in law, with the solution to (1.11). The details of the two steps 1 and 2 are similar to the procedure followed in [10, Section 6.6]

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